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NUMERICAL SIMULATION OF MICROPHYSICAL PROCESSES IN CUMULONIMBUS——PART I: MICROPHYSICAL MODEL

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ABSTRACT

A comprehensive parameterized model of microphysical processes in cumulonimbus clouds is presented. The transformation rates of the water contents and number concentrations of the cloud droplets, rain drops, ice crystals, graupels and hails are deduced on theoretical and experimental results for 26 kinds of microphysical processes, which include condensation, deposition, evaporation, collection, ice nucleation, ice multiplication, melting, freezing, and autoconversions of cloud to rain, ice to graupel and graupel to hail.

I. INTRODUCTION

Cumulonimbus is one of the major precipitation clouds in China, which sometimes even brings hails or torrential rains. The precipitation mechanism for cumulonimbus is very complicated due to its wide range of temperature, strong updraft and different kinds of hydrometeors. It is necessary to establish a numerical model including all the main microphysical processes for studying the precipitation physics and the possibility of weather modification. The parameterized models proposed by Wisner (1972), Orville (1977) and Cotton (1982) are widely used. In their usual form, the number concentrations of various water particles are diagnosed from their mass contents and the parameter N_0 for the size distributions is assumed to be constant. This assumption does not agree with observations that N_0 varies widely and may suddenly change during a particular rainfall (Pruppacher, 1978a). Cloud model of this type (one-parameter model) causes serious errors in the simulation of some important microporcesses. For example, the ice nucleation and multiplication have small effects on ice mass content, but make the ice number increase significantly. The freezing of raindrops through the collection of ice particles depends mainly on the ice number concentrations but not the mass content. These processes can hardly be simulated by a model in which the ice number concentration can not be predicted. It is widely accepted that the number concentration of ice, rain embryos and hail embryos have a considerable effect on the precipitation efficiency based on Bergeron theory, and that the artificial increase in number concentrations (not mass content) of ice and large cloud droplets is also the basis of weather modification. Therefore the prediction of number concentrations of various kinds of water particles is essential in cloud modelling. Parameterized models which predict both mass contents and number concentrations of various water particles (two-parameter model) have been established by Hu (1983, 1984, 1985 and 1986) for stratiform cluods and warm cumulus clouds. By use of these models, microprocesses in the middle latitute cyclone cloud systems and precipitation enhancement experiments have been simulated fairly successfully. Recently, Xu (1985) has established a two-parameter model of hail cloud, predicting the number concentrations of rain and hail. He has shown that the two-parameter model simulates the melting, evaporation and other processes more properly due to the conservation of the mean diameter of water particles during advection.

In this paper a comprehensive two-parameter model is presented to simulate 26 kinds of microprocesses in cumulonimbus clouds. The equations of transformation rates of mass contents and number concentrations of cloud droplets, raindrops, ice, graupels and hails are deduced for these processes. Some typical examples of modelling results will be presented in part II of this work.

II. PARAMETERIZATION OF MICROSTRUCTURES

Based on its phase and dimension, water substance can be divided into 6 categories in this model, i. e., vapor (v), cloud droplets (c), rain drops (r), ice crystals (i), graupels (g) and hails (h). A cloud droplet is defined as a liquid water drop with a diameter less than $D_{*r}=0.02$ cm. A raindrop is a larger water drop. Graupel and hail are of ice ball, they grow mainly by the collection of water drops. Hailstone has the diameter larger than $D_{*h}=0.5$ cm as defined in meteorological observation. Ice crystal grows mainly by sublimation and maintains its crystal habit. All categories of water particles are described by their number concentrations (N) and mass contents (Q), which are the total number and mass in the unit mass of air respectively. The number concentration of cloud droplets varies relatively little in a cloud, thus a constant value can be assumed for a given cloud. The model can predict 10 microphysical variables including mass contents of vapor and cloud droplets Q_v and Q_c, mass contents and number concentrations of rain, ice, graupel and hail Q_r, N_r, Q_i, N_i, Q_g, N_g, Q_h and N_h.

Every category of water particle is assumed to have a size distribution in the form

$$N(D)=N_0 D^a \exp(-\lambda D), \tag{1}$$

where $N(D)\,dD$ is the number of particles in the diameter range dD centered at D, and a, N_0 and λ are all parameters.

1. *Size Distribution of Cloud Droplets*

The Khrigian-Mazin distribution is assumed for cloud droplets ($a=2$).

$$N_c=\int_0^\infty N_0 D^2 \exp(-\lambda D)\,dD=\Gamma(3)\,N_0\lambda_c^{-3}, \tag{2}$$

$$Q_c=\int_0^\infty N_0 D^2 \exp(-\lambda D)\,\frac{\pi}{6}D^3\rho_w dD=\Gamma(6)\,\frac{\pi}{6}\rho_w N_0\lambda^{-6}=10\pi\rho_w N_c\lambda_c^{-3}, \tag{3}$$

$$\bar{D}_c\approx\frac{6}{\pi\rho_w}\left(\frac{Q_c}{N_c}\right)^{1/3}, \tag{4}$$

where $\bar{D}_c$ is the mass-mean diameter, ρ_w is the density of liquid water, $\Gamma(x)$ is the Gamma function. The fall speed of cloud droplets is neglected.

2. *Size Distribution of Raindrops and Graupels*

The Marshall-Palmer distribution is taken for raindrops ($a=0$), but with parameter N_0 variable. The same distribution is also taken for graupels based on the observation (Pruppacher, 1978b) as follows:

$$N_{r(g)}=\int_0^\infty N_0\exp(-\lambda D)\,dD=N_{0r(g)}\lambda_{r(g)}^{-1}, \tag{5}$$

$$Q_{r(g)}=\int_0^\infty N_0\exp(-\lambda D)A_mD^3dD=6A_{mr(g)}N_{0r(g)}\lambda_{r(g)}^{-4}, \tag{6}$$

$$\bar{D}_{r(g)}=\left(\frac{Q_r}{A_mN_r}\right)^{1/3}, \tag{7}$$

where $A_{mr}=\pi\rho_w/6=0.524g\text{cm}^{-3}$, A_{mg} is taken to be 0.065g cm^{-3} after Mason (1971).

The fall speed of raindrops (V_r) with diameter D_r is taken after Orville (1977) to be

$$V_r=A_{vr}\cdot D_r^{0.8}, \tag{8}$$

and V_r is limited to a value of 9.7 m/s under standard conditions(the pressure $P=P_0=10^5$ Pa). The fall speed of graupel (V_g) is taken as the function of its diameter (D_g) based on the observation of Locatelli and Hobbs (see Pruppacher, 1978c):

$$V_g=A_{vg}\cdot D_g^{0.8}. \tag{9}$$

The fall speed of particles in air also depends on the air density and temperature. In the real atmosphere it depends mainly on the air pressure (P) and $V=V_0(P/P_0)^{a_1}$ is assumed based on the calculated results of Beard (Pruppacher, 1978d). Thus

$$V_{r(g)}=A_{vr(g)}D_{r(g)}^{0.8}\left(\frac{P_0}{P}\right)^{a_1}, \tag{10}$$

where $A_{vr}=2100\text{cm}^{0.2}\text{s}^{-1}$, $A_{vg}=500\text{cm}^{0.2}\text{s}^{-1}$, and $a_1=0.286$. The mass-averaged fall speed (V) is deduced as

$$\bar{V}_{r(g)}=\frac{1}{Q}\int_0^\infty N_0\exp(-\lambda D)A_vD^{0.8}A_mD^3\left(\frac{P_0}{P}\right)^{a_1}dD$$

$$=\frac{\Gamma(4.8)}{\Gamma 4}A_{vr(g)}\lambda_{r(g)}^{-0.8}\left(\frac{P_0}{P}\right)^{a_1}. \tag{11}$$

3. *Size Distribution of Ice Crystals*

Based on observations a is taken to be 1, and mass and fall speed of single ice is taken to be in proportion to D_i^2 and $D_i^{0.3}$ respectively after Hu (1986). Therefore

$$N_i=\int_0^\infty N_0D\exp(-\lambda D)\,dD=N_{0i}\lambda_i^{-2}, \tag{12}$$

$$Q_i=\int_0^\infty N_0D\exp(-\lambda D)A_{mi}D^2dD=6N_{0i}A_{mi}\lambda_i^{-4}, \tag{13}$$

$$\bar{D}_i=\left(\frac{Q_i}{A_{mi}N_i}\right)^{1/2}=2.88\lambda_i^{-1}, \tag{14}$$

$$V_i=A_{vi}D_i^{\frac{1}{3}}\left(\frac{P_0}{P}\right)^{a_2}, \tag{15}$$

$$M_i=A_{mi}D_i^2, \tag{16}$$

$$\bar{V}_i=\frac{1}{Q}\int_0^\infty N_0D_i\exp(-\lambda D_i)A_{mi}D_i^2A_{vi}D_i^{\frac{1}{3}}\left(\frac{P_0}{P}\right)^{a_2}dD_i$$

$$=\frac{\Gamma(4.33)}{\Gamma(4)}A_{vi}\lambda_i^{-\frac{1}{3}}\left(\frac{P_0}{P}\right)^{a_2}, \tag{17}$$

where $A_{mi}=0.001g\text{cm}^{-?}$, $A_{vi}=70\text{cm}^{\frac{2}{3}}\text{s}^{-1}$ and $a_2=0.3$.

4. *Size Distribution of Hailstones*

The truncated Gamma function is taken for the size distribution of hailstones due to the large value of their minimum diameter ($D_{*h}=0.5$ cm)

$$\begin{cases} N(D)=N_0\exp(-\lambda D), & D\geqslant D_{*h} \\ N(D)=0 & D<D_{*h} \end{cases} \tag{18}$$

The fall speed of single hailstone is taken to be in proportion to $D^{0.8}$ after Auer (1972), and corrected by the air density (ρ_0). (see Pruppacher, 1978e), then

$$V_h=A_{vh}D_h^{0.8}\left(\frac{\rho_0}{\rho}\right)^{\frac{1}{2}}, \tag{19}$$

where $A_{vh}=810\ \mathrm{cm}^{0.2}\mathrm{s}^{-1}$ (for $\rho_0=1.27\ \mathrm{kg\ m}^{-3}$),

$$N_h=\int_{D_*}^{\infty}N_{0h}\exp(-\lambda_h D)dD=N_{0h}\exp(-\lambda_h D_*)\lambda_h^{-1}, \tag{20}$$

$$Q_h=\int_{D_*}^{\infty}N_0\exp(-\lambda_h D)A_{mh}D^3dD$$

$$=6N_h\lambda_h^{-3}A_{mh}\left[1+\lambda_h D_*+\frac{1}{2}(\lambda_h D_*)^2+\frac{1}{6}(\lambda_h D_*)^3\right], \tag{21}$$

$$\lambda_h^{-1}=\sqrt[3]{\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+p^3}}+\sqrt[3]{\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+p^3}}-\frac{D_*}{3}, \tag{22}$$

$$\lambda_h^{-1}\approx\sqrt[3]{q}+p\,\frac{1}{\sqrt[3]{q}}-\frac{D_*}{3}=\sqrt[3]{q}+\frac{D_*^2}{18\sqrt[3]{q}}-\frac{D_*}{3}, \tag{23}$$

where

$$q=-\frac{2D_*^3}{27}+\frac{Q_h}{6A_{mh}N_h},\qquad p=\frac{D_*^2}{18},$$

$$\bar{V}_h=\int_{D_*}^{\infty}A_{vh}\exp(-\lambda_h D)D^{3.8}dD\Big/\int_{D_*}^{\infty}\exp(-\lambda_h D)D^3dD\sqrt{\frac{\rho_0}{\rho}},$$

$$\bar{V}_h\approx\sqrt{\frac{\rho_0}{\rho}}A_{vh}\lambda_h^{-0.8}\Big[\beta_h^{3.8}+3.8\beta_h^{2.8}+10.64\beta_h^{1.8}$$

$$+19.5\int_{D_*}^{\infty}\exp(-\lambda D)D^{0.8}dD\Big]\Big/[\beta_h^3+3\beta_h^2+6\beta_h+6], \tag{24}$$

$$\bar{V}_h\approx\sqrt{\frac{\rho_0}{\rho}}A_{vh}\lambda_h^{-0.8}[\beta_h^{3.8}+3.8\beta_h^{2.8}+10.64\beta_h^{1.8}+17.84(0.8\beta_h+1)]/$$
$$[\beta_h^3+3\beta_h^2+6\beta_h+6], \tag{24a}$$

$$\bar{D}_h=\left(\frac{Q_h}{A_{mh}N_h}\right)^{1/3}=[\beta_h^3+3\beta_h^2+6\beta_h+6]^{1/3}\lambda_h^{-1}, \tag{25}$$

where $\beta_h=\lambda_h D_*$, $D_*=0.5$ cm, $A_{mh}=0.471\ \mathrm{g\ cm}^{-3}$, and $A_{vh}=810\ \mathrm{cm}^{0.2}\mathrm{s}^{-2}$.

The results of numerical integration of Eq. (24) show that the error from (24a) is less than 2%.

III. EQUATIONS OF CLOUD MICROPHYSICS

The prognostic equation of each microphysical variable (M) in a one-dimensional cloud model can be expressed as

$$\frac{\partial M}{\partial t}=-(W-V_m)\frac{\partial M}{\partial z}+k\frac{\partial^2 M}{\partial z^2}-E(M-M_e)+\frac{M}{\rho}\frac{\partial\rho V_m}{\partial z}+\frac{\delta M}{\delta t}. \tag{26}$$

The terms in the right hand of Eq. (26) describe the effects of advection, turbulent mixing, entrainment, convergence due to falling and source. The source term includes the effects of all related microphysical processes. 26 kinds of microprocesses are considered in this model, such as condensation (deposition or evaporation)of cloud droplet, rain, ice, graupel and hail ($S_{vc}, S_{vr}, S_{vi}, S_{vg}, S_{vh}$), collection of cloud droplets by rain, ice, graupel and hail (C_{cr}, C_{ci}, C_{cg}, C_{ch}), collection of rain by ice, graupel and hail (C_{ri}, C_{rg}, C_{rh}), collection of ice by rain, graupel and hail (C_{ir}, C_{ig}, C_{ih}), collisions of ice with ice and rain with rain (C_{ii}, C_{rr}), ice nucleation and multiplication (P_{vi}, P_{ci}), autoconversions of cloud to rain, ice to graupel and graupel to hail (A_{cr}, A_{ig}, A_{gh}), self-freezing of raindrop (M_{rg}), and melting of ice, graupel and hail (M_{ir}, M_{gr}, M_{hr}). All microprocesses are denoted by 3 English letters, the first capital letter expresses the name of transformation process, the second and third small letters express the dissipation and production phases of water substance, respectively. They are also used to express the transformation rate of mass content. These 3 letters with additional letter N in the front are used to express the change rate of number concentration for the given process. For example, C_{ir} expresses the collection of ice by rain. If $T>T_0$, then

$$C_{ir}=-\frac{\delta Q_i}{\delta t}=\frac{\delta Q_r}{\delta t}, \quad NC_{ir}=-\frac{\delta N_i}{\delta t}, \ \frac{\delta N_r}{\delta t}=0 ,$$

If $T<T_0$, then

$$\frac{\delta Q_i}{\delta t}=-C_{ir}, \frac{\delta Q_r}{\delta t}=-C_{ri}, \frac{\delta Q_g}{\delta t}=C_{ir}+C_{ri}, \frac{\delta N_g}{\delta t}=-\frac{\delta N_i}{\delta t}=-\frac{\delta N_r}{\delta t}=NC_{ir},$$

because ice transforms into graupel after its collision with rain.

The source term of each microphysical variable can be expressed as

$$\frac{\delta Q_v}{\delta t}=-S_{vc}-S_{vr}-S_{vi}-S_{vg}-S_{vh}-NP_{vi}Q_{vo} , \tag{28}$$

$$\frac{\delta Q_c}{\delta t}=S_{vc}-C_{ci}-C_{cr}-C_{cg}-C_{ch}-A_{cr}-NP_{ci}Q_{co}, \tag{29}$$

$$\left\{\begin{array}{lll} \frac{\varrho Q_r}{\delta t}=S_{vr}+C_{cr}+A_{cr}+M_{gr}+M_{ir}+M_{kr}-M_{rg} & & \\ \text{If} \quad T\geqslant 273\text{K}, & +C_{ch}+C_{ir}; & \\ \text{if} \quad T<273\text{K}, & -C_{rh}-C_{rg}-C_{ri}; & (kk=0) \\ & -C_{wh}+C_{ch}-C_{rg}-C_{ri}, & (kk=1) \end{array}\right. \tag{30}$$

$$\left\{\begin{array}{ll} \frac{\delta N_r}{\delta t}=NS_{vr}+A_{cr}/Q_{ro}+NM_{gr}+NM_{ir}-NM_{rg}+NC_{rr} & \\ \text{If} \quad T\geqslant 273\text{K}, & (M_{hr}+C_{ch})/Q_{ho}; \\ \text{if} \quad T<273\text{K}, & -NC_{rh}-NC_{rg}-NC_{ri}; \quad (kk=0) \\ & -NC_{rh}-NC_{rg}-NC_{ri}-(C_{ch}+C_{rh}-C_{wh})/Q_{ho}, (kk=1) \end{array}\right. \tag{31}$$

$$\frac{\delta Q_i}{\delta t}=S_{vi}+C_{ci}-C_{ir}-M_{ir}-C_{ig}-C_{ih}-A_{ig}+NP_{vi}Q_{vo}+NP_{ci}Q_{co}, \tag{32}$$

$$\frac{\delta N_i}{\delta t}=NS_{vi}-NC_{ir}-NM_{ir}-NC_{ig}-NC_{ih}-NA_{ig}+NP_{vi}+NP_{ci}+NC_{ii}, \tag{33}$$

$$\left\{\begin{array}{ll} \frac{\delta Q_g}{\delta t}=S_{vg}+C_{cg}+C_{ig}-C_{gh}+A_{ig}+M_{rg}-M_{gr}-A_{gh} & \\ \text{If} \quad T<273\text{K}, & +C_{rg}+C_{ir}+C_{ri}, \end{array}\right. \tag{34}$$

$$\left\{\begin{array}{ll} \frac{\delta N_g}{\delta t}=NS_{vg}+NA_{ig}-NA_{gh}+NM_{rg}-NM_{gr}-NC_{gh} & \\ \text{If} \quad T<273\text{K}, & +N_{cir} , \end{array}\right. \tag{35}$$

$$\begin{cases} \dfrac{\delta Q_h}{\delta t} = S_{vh} + C_{ih} + C_{gh} - M_{hr} + A_{gh}, \\ \text{if} \quad T < 273\text{K}, \qquad + C_{ch} + C_{rh}, \quad (kk=0) \\ \qquad\qquad\qquad\qquad + C_{wh} \quad , \qquad (kk=1) \end{cases} \tag{36}$$

$$\frac{\delta N_h}{\delta t} = N A_{gh} - N M_{hr}. \tag{37}$$

where C_{wh} is the threshold value, which determines the growth regime of hail. If the collection rate of supercooled water by hail $(C_{ch}+C_{rh})$ is less than C_{wh}, then hail grows in the dry growth regime $(kk=0)$. All collected water will freeze on the hail. If $C_{ch}+C_{rh} \geqslant C_{wh}$, then hail grows in the wet growth regime $(kk=1)$, only part of the collected water $(C_{wh} \cdot dt)$ will freeze on the hail, the remainder dt. $(C_{rh}+C_{ch}-C_{wh})$ will form a liquid water film on the hail and later the film will shed from hail and transform into secondary raindrops. The average diameter of these secondary raindrops is about 1.4 mm according to the experiments of Joe and List (1980). The corresponding mass Q_{ho} is about 1.47 10^{-3} g. The diameter of newly-formed raindrops by cloud-rain autoconversion is taken to be 0.2 mm according to its defination, and the corresponding mass (Q_{ro}) is about 4.19×10^{-6} g. Average mass of the newly- nucleated ice (Q_{vo}) is taken to be 10^{-10} g. Average mass of the newly-multiplicated ice (Q_{co}) is taken to be 10^{-9} g.

IV. EQUATIONS OF CONDENSATION, EVAPORATION AND DEPOSITION

1. *Condensation, Evaporation and Deposition of Cloud Droplets and Ice Crystals*

In a parameterized cloud model the supersaturation of vapor in cloud is usually ignored for the simplification of the condensation equation. But it causes significant errors in simulation of the Bergeron process when 3 phases of water coexist. In this model humidity of the air is predicted and the condensation is calculated based on the supersaturation of the vapor:

$$S_{vc} = \int_0^\infty N_0 D^2 \exp(-\lambda D)\, 2\pi k_d \rho D \left[1 + \frac{L_V k_d \rho Q_{SW}}{k_T T}\left(\frac{L_V}{RT} - 1\right)\right]^{-1} (Q_V - Q_{SW})\, dD$$
$$= A_{VC}(Q_V - Q_{SW}), \tag{38}$$

where

$$A_{VC} = 6\pi k_d \rho \left[1 + \frac{L k_d \rho Q_{SW}}{k_T T}\left(\frac{L_V}{RT} - 1\right)\right]^{-1} N_c (10\pi N_c / Q_c)^{\frac{1}{3}}.$$

The deposition rate of ice crystal (dm_i/dt) in a water-saturated environment under different temperatures is expressed by Koenig (1972)

$$\frac{\delta m_i}{\delta t} = a_1 \cdot m_i^{a_2},$$

where a_1 and a_2 are parameters, whose values are given for different temperatures. The deposition rate of ice population in environments with any humidity (Q_v) and temperature can then be deduced

$$S_{Vi} = \int_0^\infty N_0 D \exp(-\lambda D)\, a_1 (A_{mi} D^2)^{a_2} \frac{(Q_V - Q_{Si})}{(Q_{SW} - Q_{Si})} dD = A_{Vi}(Q_V - Q_{Si}), \tag{39}$$

where

$$A_{Vi} = 2a_1 N_i (6N_i/Q_i)^{-a_2} (Q_{SW} - Q_{Si})^{-1},$$

$$NS_{Vi}\begin{cases} =0 , & S_{Vi}\geqslant 0 \\ =S_{Vi}\cdot N_i/Q_i. & S_{Vi}<0 \end{cases} \tag{39a}$$

Condensation is a source term of the vapor, then

$$\frac{\delta Q_V}{\delta t}=-S_{VC}=-A_{VC}(Q_V-Q_{SW}) .$$

Considering the influences of the latent heat of condensation on the air temperature and saturated humidity, we can get

$$\frac{\delta(Q_V-Q_{SW})}{\delta t}=-A_{VC}(Q_V-Q_{SW})\left(1+\frac{\in L^2Q_{SW}}{C_pRT^2}\right), \tag{40}$$

The relax time of condensation t_* can be deduced

$$t_*=\left[A_{VC}\left(1+\frac{\in L^2Q_{SW}}{C_pRT^2}\right)\right]^{-1}, \tag{40a}$$

which is only about 1 s due to the very large number concentration of cloud droplets ($N_C=10^8$ kg^{-1}). Therefore the time step for calculation of Eq. (38) must be less than 1 s. The relax time for other microprocesses is much longer and a larger time step can be used. Consequently, the processes of condensation of cloud droplets and ice crystals are calculated with smaller time steps (Dt_s) and all the other processes including advection, turbulent, etc., are calculated with larger time steps (Dt) for saving computer time. In this way, vapor and temperature are calculated with a smaller time step (Dt_s), which is taken to be $(1/N_s)Dt$

$$\begin{cases} Q_V^{t+\Delta t}=Q_V^t+[B_V-A_{Vi}(Q_V-Q_{Si})^t-A_{VC}(Q_V-Q_{SW})^t]Dt_S, \\ T^{t+\Delta t}=T^t+\left[B_T+\frac{L_S}{C_P}A_{Vi}(Q_V-Q_{Si})^t+\frac{L_V}{C_P}A_{VC}(Q_V-Q_{SW})^t\right]Dt_s, \end{cases} \tag{41}$$

where $\Delta t=Dt_S$, and B_V and B_T are the total change rate of the vapor and temperature due to all macro- and micro-processes except the condensation of cloud and ice. B_V, B_T, A_{Vi} and A_{VC} are calculated with larger time steps, and keep unchanged in the given large time step. After calculating Eq. (41) with a smaller time-step for Ns times, a larger time-step is complished, and the average value of S_{Vc} and S_{Vi} are utilized in the computation of the cloud droplets and ice

$$\begin{cases} S_{VC}=\frac{1}{Dt}\sum^{Ns}A_{VC}(Q_V-Q_{SW})^tDt_S, \\ S_{Vi}=\frac{1}{Dt}\sum^{Ns}A_{Vi}(Q_V-Q_{Si})^tDt_S, \end{cases} \tag{41a}$$

The calculation with this type of program is stable and economical.

2. *Sublimation of Graupels*

The sublimation rate of a single graupel dm/dt can be expressed as

$$\frac{dm}{dt}=2\pi Dkd\rho(Q_V-Q_d)(1+0.23Re^{\frac{1}{2}}), \tag{42}$$

where D is the diameter of graupel, K_d is the diffusivity of water vapor into air, Q_d and T_d are the mixing ratio (of vapor) and the temperature near the surface of the graupel. When the graupel grows in the wet growth regime, $Q_d=Q_{S0}$, $T_d=T_0$ where T_0 and Q_{S0} are the ice melting temperature (273 K) and the corresponding saturated vapor mixing ratio. Then

$$\left.\frac{dm}{dt}\right|_{W}=2\pi Dk_{d}\rho(Q_{V}-Q_{S0})\,(1+0.23Re^{\frac{1}{2}}). \tag{43}$$

When the graupel grows in the dry growth regime,

$$\frac{Q_S-Q_d}{Q_S}=\frac{Q_S(T)-Q_S(T_d)}{Q_S(T)}\approx\frac{T-T_d}{T}\left(\frac{L}{RT}-1\right). \tag{44}$$

The equation of heat balance can be expressed as

$$L_S\frac{dm}{dt}-2\pi Dk_t(T-T_d)\,(1+0.23Re^{\frac{1}{2}})+L_f(C_{cg}+C_{rg})=0\,. \tag{45}$$

From Eqs. (42), (44) and (45) we can get

$$\left.\frac{dm}{dt}\right|_{d}=[2\pi Dk_d\rho\,(Q_V-Q_S)\,(1+0.23Re^{\frac{1}{2}})-\frac{L_fk_d\rho Q_S}{k_tT}\left(\frac{L}{RT}-1\right)(C_{cg}+C_{rg})$$
$$\times\left[1+\frac{L_sk_d\rho Q_S}{k_tT}\left(\frac{L}{RT}-1\right)\right]^{-1}. \tag{46}$$

The total sublimation rate of the graupel population can also be deduced for wet and dry growth regimes as

$$S_{Vgw}=\int_0^\infty 2\pi k_d\rho(Q_V-Q_{S0})\,N_0\exp(-\lambda D)\,D\left(1+0.23\sqrt{\frac{\rho A_{Vg}}{\mu}}\,D^{0.9}\right)dD$$
$$=2\pi k_d\rho(Q_v-Q_{S0})\,N_g\,(6A_{mg}N_g/Q_g)^{-1/3}$$
$$\times\left[1+0.23\sqrt{\frac{\rho A_{vg}}{\mu}}\Gamma(2.9)\,(6A_{mg}N_g/Q_g)^{0.3}\right], \tag{47}$$

$$S_{vgd}=\left\{\int_0^\infty 2\pi k_d\rho\,(Q_V-Q_{Si})\,N_0\exp(-\lambda D)\;D\left(1+0.23\sqrt{\frac{\rho A_{Vg}}{\mu}}D^{0.9}\right)dD\right.$$
$$\left.-\frac{L_fk_d\rho Q_{Si}}{k_tT}\left(\frac{L_S}{RT}-1\right)(C_{vg}+C_{rg})\right\}\left[1+\frac{L_Sk_D\rho Q_{Si}}{k_TT}\left(\frac{L_S}{RT}-1\right)\right]^{-1}$$
$$=\left\{2\pi k_d\rho\,(Q_V-Q_{Si})\,N_g\,(6A_{mg}N_g/Q_g)^{-1/3}\left[1+0.23\sqrt{\frac{\rho A_{Vg}}{\mu}}\Gamma(2.9)\right.\right.$$
$$\left.\left.\times\,(6A_{mg}N_g/Q_g)^{0.3}\right]-\frac{L_fk_d\rho Q_{Si}}{k_tT}\left(\frac{L_S}{RT}-1\right)(C_{cg}+C_{rg})\right\}$$
$$\times\left[1+\frac{L_Sk_d\rho Q_{Si}}{k_tT}\left(\frac{L_S}{RT}-1\right)\right]^{-1}, \tag{48}$$

$$S_{Vg}\begin{cases}=MAX[S_{vgw},S_{Vgd}], & T<T_0\\ =S_{Vgw}. & T\geqslant T_0\end{cases} \tag{49}$$

The sublimation (or condensation) has no effect on the number concentration but the evaporation would decrease it. The evaporation rate (dm/dt) is approximately in proportion to $D^{1.9}$, thus the change rate of diameter (dD/dt) is approximately in proportion to $D^{0.1}$. It means that parameter λ of the size distribution changes little in evaporation, so that $dN_g/N_g\approx dQ_g/Q_g$ and

$$NS_{Vg}\begin{cases}=0\,, & S_{Vg}\geqslant 0\\ =S_{Vg}N_g/Q_g. & S_{Vg}<0\end{cases} \tag{50}$$

3. *Condensation (Evaporation) of Raindrops*

$$S_{Vr}=\int_0^\infty 2\pi k_d\rho(Q_V-Q_{SW})N_0\exp(-\lambda D)D\left[1+0.23\sqrt{\frac{\rho A_{Vr}}{\mu}}D^{0.9}\right]dD$$

$$\times\left[1+\frac{L_Vk_d\rho Q_{SW}}{k_tT}\left(\frac{L_V}{RT}-1\right)\right]^{-1}$$

$$=2\pi k_d\rho(Q_V-Q_{SW})N_r(6A_{mr}N_r/Q_r)^{-1/3}\left[1+0.23\Gamma(2.9)\sqrt{\frac{\rho A_{Vr}}{\mu}}\right.$$

$$\left.\times(6A_{mr}N_r/Q_r)^{-0.3}\right]\left[1+\frac{L_Vk_d\rho Q_{SW}}{k_tT}\left(\frac{L_V}{RT}-1\right)\right]^{-1}, \tag{51}$$

$$NS_{Vr}\begin{cases}=0, & S_{Vr}\geqslant 0\\ S_{Vr}N_r/Q_r, & S_{Vr}<0\end{cases} \tag{52}$$

where L_f, L_V and L_S are the latent heat of melting, evaporation and sublimation respectively, k_t is the heat conductivity of air, μ is the dynamic viscosity of air.

4. *Sublimation of Hail*

The sublimation growth equation of hailstone population can be deduced for different growth regimes with $1+0.29R_e^{1/2}\approx 0.29R_e^{1/2}$ in mind.

In wet growth regime $(kk=1)$

$$S_{Vh}=2\pi k_d\rho(Q_V-Q_{S0})0.29\sqrt{\frac{\rho A_{Vh}}{\mu}}\int_{D_*}^\infty N_0D^{1.9}\exp(-\lambda_hD)dD$$

$$\approx 2\pi k_d\rho(Q_V-Q_{S0})0.29\sqrt{\frac{\rho A_{Vh}}{\mu}}N_h\lambda_h^{-1.9}[(\lambda_hD_*)^{1.9}+\Gamma(2.9)$$

$$\times(0.9\lambda_hD_*+1)]. \tag{54}$$

In dry growth regime $(kk=0)$

$$S_{Vh}=\left[\int_{D_*}^\infty 2\pi k_d\rho(Q_V-Q_{Si})0.29\sqrt{\frac{\rho A_{Vh}}{\mu}}D^{0.9}N_0\exp(-\lambda D)dD\right.$$

$$\left.-\frac{L_fk_d\rho Q_{Si}}{k_tT}\left(\frac{L_S}{RT}-1\right)(C_{ch}+C_{rh})\right]\left[1+\frac{L_Sk_d\rho Q_{Si}}{k_tT}\left(\frac{L_S}{RT}-1\right)\right]^{-1}$$

$$=\left\{2\pi k_d\rho(Q_V-Q_{Si})0.29\sqrt{\frac{\rho A_{Vh}}{\mu}}N_h\lambda_h^{-1.9}[(\lambda_hD_*)^{1.9}+\Gamma(2.9)\right.$$

$$\left.\times(0.9\lambda_hD_*+1)]-\frac{L_fk_d\rho Q_{Si}}{k_tT}\left(\frac{L_S}{RT}-1\right)(C_{ch}+C_{rh})\right\}\left[1+\frac{L_Sk_d\rho Q_{Si}}{k_tT}\right.$$

$$\left.\times\left(\frac{L_S}{RT}-1\right)\right]^{-1}. \tag{55}$$

V. EQUATIONS OF COLLECTION PROCESSES

1. *Collection of Cloud Droplets by Ice*

The theoretical and observational results, summarized by Pruppacher (1978f) show that ice crystals can collect only the cloud droplets greater than 15 μm in diameter and the collection efficiency averaged for all larger cloud droplets ($\bar{E}_{ci}$) depends on the diameter of ice (D_i). The following values of E are taken.

$$\bar{E}_{ci}\begin{cases} =0 & D_i<0.03\text{cm} \\ =15(D_i-0.03) & 0.03<D_i<0.05\text{cm} \\ =0.3+10(D_i-0.05) & 0.05<D_i<0.07\text{cm} \\ =0.5+5(D_i-0.07) & 0.07<D_i<0.11\text{cm} \\ =0.7 & D_i>0.11\text{cm} \end{cases}, \tag{56}$$

$$C_{ci}=\int_0^\infty\int_{D_i^*}^\infty \pi D_i^2 A_{Vi} D_i^{1/3} N_{0i}\exp(-\lambda_i D_i) D_i \rho\frac{\pi}{6} D_C^3 N_{0c}\exp(-\lambda_c D_c) D_C^2 \bar{E}_{ci} dD_c dD_i$$

$$=\frac{\pi}{4}\Gamma\left(4\frac{1}{3}\right)A_{Vi}\rho Q_0\bar{E}_{ci}(6A_{mi}N_i/Q_i)^{-7/6}N_i\exp(-\beta_1)\left[1+\sum_{i=1}^{5}\frac{\beta_1^i}{i!}\right], \tag{57}$$

where $\beta_1=\lambda_0 D_1^*, \lambda_0=(10\rho_W\pi N_c/Q_c)^{1/3}$, $D_1^*=15\mu m$. $\bar{E}_{ci}$ is the collection coefficient averaged for all ice crystals, which can be obtained from Eq. (56) by assuming that the corresponding ice diameter (D_i) equals the mass-median diameter of ice population $D_{io}=3.67/\lambda_i$.

2. *Collection of Cloud Droplets by Rain and Graupels*

$$C_{cr}=\int_0^\infty N_o\exp(-\lambda D)\pi D^2 A_{Vr} D^{0.3}\rho Q_c E\left(\frac{P_0}{P}\right)^{a_1} dD$$

$$=\frac{\pi}{4}\Gamma(3.8)A_{Vr}\rho Q_c\bar{E}N_r[6A_{mr}N_r/Q_r]^{-2.8/3}\left(\frac{P_0}{P}\right)^{a_1}, \tag{58}$$

$$C_{cg}=\int_0^\infty N_0\exp(-\lambda D)\pi D^2 A_{Vg} D^{0.3}\rho Q_c E\left(\frac{P_0}{P}\right)^{a_1} dD$$

$$=\frac{\pi}{4}\Gamma(3.8)[A_{mg}6]^{-1}\rho Q_c\bar{E}Q_g[6A_{mg}N_g/Q_g]^{0.2/3}\left(\frac{P_0}{P}\right)^{a_1}A_{vg}. \tag{59}$$

3. *Collection of Cloud Droplets by Hail*

$$C_{ch}=\frac{\pi}{4}A_{Vh}\rho Q_c E\int_{D_*}^\infty N_{0h}\exp(-\lambda_h D)D^{2.8}dD\left(\frac{\rho_0}{\rho}\right)^{1/2}$$

$$\approx\frac{\pi}{4}A_{Vh}\rho Q_c E N_h\lambda^{-2.8}[(\lambda_h D_*)^{2.8}+2.8(\lambda_h D_*)^{1.8}+\Gamma(3.8)$$

$$\times(0.8\lambda_h D_*+1)]\left(\frac{\rho_0}{\rho}\right)^{1/2}. \tag{60}$$

4. *Collection of Ice by Rain*

$$C_{ir}=\frac{\pi}{4}\rho A_{mi}(\bar{V}_r-\bar{V}_i)\bar{E}_{ri}\int_0^\infty\int_0^\infty N_{0r}N_{0i}(D_r+D_i)^2 D_i^3\exp(-\lambda_i D_i)$$

$$\times\exp(-\lambda_r D_r)dD_i dD_r$$

$$=\frac{\pi\rho}{12A_{mr}}Q_iQ_r\lambda_r\left[1+4\frac{\lambda_r}{\lambda_i}+10\left(\frac{\lambda_r}{\lambda_i}\right)^2\right]|\bar{V}_r-\bar{V}_i|\bar{E}_{ri}, \tag{61}$$

$$NC_{ir}=\frac{\pi}{4}\rho(\bar{V}_r-\bar{V}_i)E_{ri}\int_0^\infty\int_0^\infty N_{0r}N_{0i}(D_r+D_i)^2D_i\exp(-\lambda_iD_i)$$
$$\times\exp(-\lambda_rD_r)dD_idD_r$$
$$=\frac{\pi}{12A_{mr}}\rho N_iQ_r\lambda_r\left[1+2\frac{\lambda_r}{\lambda_i}+3\left(\frac{\lambda_r}{\lambda_i}\right)^2\right]|\bar{V}_r-\bar{V}_i|E_{ri}, \tag{62}$$

where E_{ri} is the averaged collection coefficient of ice-rain, which has not been well studied and $\bar{E}_{ri}=0.8$ is taken.

5. *Collection of Rain by Ice*

$$C_{ri}=\frac{\pi}{4}\rho A_{mr}|\bar{V}_r-\bar{V}_i|\bar{E}_{ri}\int_0^\infty\int_0^\infty N_{0r}N_{0i}(D_r+D_i)^2D_iD_r^3\exp(-\lambda_iD_i)$$
$$\times\exp(-\lambda_rD_r)dD_idD_r$$
$$=5\pi N_i\rho Q_r\lambda_r^{-2}\left[1+0.8\left(\frac{\lambda_r}{\lambda_i}\right)+0.3\left(\frac{\lambda_r}{\lambda_i}\right)^2\right]\bar{E}_{ri}|\bar{V}_r-\bar{V}_i|, \tag{63}$$

$$NC_{ri}=NC_{ir}=\frac{\pi}{4}\rho|\bar{V}_r-\bar{V}_i|\bar{E}_{ri}\int_0^\infty\int_0^\infty N_{0r}N_{0i}D_i(D_r+D_i)^2\exp(-\lambda_iD_i)$$
$$\times\exp(-\lambda_rD_r)dD_idD_r$$
$$=C_{ri}N_r/Q_r\left[1+4\left(\frac{\lambda_r}{\lambda_i}\right)+10\left(\frac{\lambda_r}{\lambda_i}\right)^2\right]\left[10+8\left(\frac{\lambda_r}{\lambda_i}\right)+3\left(\frac{\lambda_r}{\lambda_i}\right)^2\right]^{-1} \tag{64}$$

6. *Collection of Rain by Graupel*

$$C_{rg}=\frac{\pi}{24}\bar{E}_{rg}A_{Vr}Q_r\rho(6A_{mr}N_r/Q_r)^{\frac{2.8}{3}}N_gKM_{rg}, \tag{65}$$

$$KM_{rg}=356.4\left[1+0.266\left(\frac{\lambda_r}{\lambda_g}\right)+0.044\left(\frac{\lambda_r}{\lambda_g}\right)^2\right]\left|1-\frac{A_{Vg}}{A_{Vr}}\left(\frac{\lambda_r}{\lambda_g}\right)^{0.8}\right|, \tag{66}$$

$$NC_{rg}=\frac{\pi}{4}A_{Vr}\bar{E}_{rg}N_r(6A_{mr}N_r/Q_r)^{-\frac{2.8}{3}}\rho N_gKN_{rg}, \tag{67}$$

$$KN_{rg}=2\times2.97\left[1+\frac{\lambda_r}{\lambda_g}+\left(\frac{\lambda_r}{\lambda_g}\right)^2\right]\left|1-\frac{A_{Vg}}{A_{Vr}}\left(\frac{\lambda_r}{\lambda_g}\right)^{0.8}\right|, \tag{68}$$

where $\bar{E}_{rg}$ is the averaged collection efficiency of rain-graupel and $\bar{E}_{rg}=0.8$ is taken.

7. *Collection of Rain by Hail*

$$C_{rh}=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_r|\int_{D*}^\infty\int_0^\infty\exp(-\lambda_rD_r-\lambda_hD_h)N_{0r}N_{0h}E(D_r+D_h)^2A_{mr}D_r^3\,dD_rdD_h$$

$$=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_r|\bar{E}_{rh}Q_rN_h\lambda_h^{-2}\left[(\lambda_hD_*)^2+2\lambda_hD_*+2+8\frac{\lambda_h}{\lambda_r}(\lambda_hD_*+1)+20\left(\frac{\lambda_h}{\lambda_r}\right)^2\right], \tag{69}$$

$$NC_{rh}=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_r|\int_{D_*}^{\infty}\int_0^{\infty}\exp(-\lambda_r D_r-\lambda_h D_h)N_{0r}N_{0h}E(D_r+D_h)^2 dD_r dD_h$$

$$=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_r|\bar{E}_{rh}N_rN_h\lambda_h^{-2}\left[(\lambda_h D_*)^2+2(\lambda_h D_*)+1+2\frac{\lambda_h}{\lambda_r}(\lambda_h D_*+1)+2\left(\frac{\lambda_h}{\lambda_r}\right)^2\right], \tag{70}$$

where $\bar{E}_{rh}=0.8$ is taken.

8. *Collection of Graupel by Hail*

$$C_{gh}=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_g|\bar{E}_{gh}Q_gN_h\lambda_h^{-2}\left[(\lambda_h D_*)^2+2\lambda_h D_*+2+8\frac{\lambda_h}{\lambda_g}(\lambda_h D_*+1)+20\left(\frac{\lambda_h}{\lambda_g}\right)^2\right], \tag{71}$$

$$NC_{gh}=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_g|\bar{E}_{gh}N_gN_h\lambda_h^{-2}\left[(\lambda_h D_*)^2+2\lambda_h D_*+1+2\frac{\lambda_h}{\lambda_g}(\lambda D_{h*}+1)+2\left(\frac{\lambda_h}{\lambda_g}\right)^2\right], \tag{72}$$

where $\bar{E}_{gh}$ is the averaged collection coefficient of graupel-hail, whose value depends on the condition of hail surface, $\bar{E}_{gh}=0.8$ is taken for wet hail ($kk=1$) and $\bar{E}_{gh}=0.1$ for dry hail ($kk=0$).

9. *Collection of Ice by Graupel*

$$C_{ig}=\int_0^{\infty}\int_0^{\infty}\pi N_{og}N_{oi}D_i(D_g+D_i)^2|\bar{V}_g-\bar{V}_i|\exp(-\lambda_i D_i-\lambda_g D_g)E_{ig}\rho Q_i dD_i dD_g$$

$$=\frac{\pi}{12A_{mg}}Q_i\rho Q_g\lambda_g\left[1+4\frac{\lambda_g}{\lambda_i}+10\left(\frac{\lambda_g}{\lambda_i}\right)^2\right]\left|\bar{V}_g-\bar{V}_i\right|\bar{E}_{ig}, \tag{73}$$

$$NC_{ig}=C_{ig}N_i/Q_i, \tag{74}$$

where $\bar{E}_{ig}=0.1$ is simply taken.

10. *Collection of Ice by Hail*

$$C_{ih}=\frac{\pi}{4}\rho\int_{D_*}^{\infty}\int_0^{\infty}N_{oi}N_{oh}\exp(-\lambda_i D_i-\lambda_h D_h)E(D_i+D_h)^2 AmiD_i^3 dD_i dD_h|\bar{V}_h-\bar{V}_i|$$

$$=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_i|\bar{E}_{ih}Q_iN_h\lambda_h^{-2}[(\lambda_h D_*)^2+2(\lambda_h D_*)+2+8\frac{\lambda_h}{\lambda_i}(\lambda_h D_*+1)$$

$$+20\left(\frac{\lambda_h}{\lambda_i}\right)^2\Big], \tag{75}$$

$$NC_{ih}=\frac{\pi}{4}\rho\int_{D_*}^{\infty}\int_0^{\infty}N_{oi}N_{oh}\exp(-\lambda_i D_i-\lambda_h D_h)E(D_i+D_h)^2 D_i dD_i dD_h|\bar{V}_h-\bar{V}_i|$$

$$=\frac{\pi}{4}\rho|\bar{V}_h-\bar{V}_i|\bar{E}_{ih}N_iN_h\lambda_h^{-2}\left[(\lambda_h D_*)^2+2(\lambda_h D_*)+2+4\frac{\lambda_h}{\lambda_i}(\lambda_h D_*+1)+6\left(\frac{\lambda_h}{\lambda_i}\right)^2\right], \tag{76}$$

where $\bar{E}_{ih}=0.1$ is taken for dry hail ($kk=0$) and $\bar{E}_{ih}=0.8$ for wet hail ($kk=1$).

11. *Collision and Breakup of Raindrops*

This process includes the collision-coalescence, collision-breakup and self-breakup, which changes the number concentration of raindrops but not their mass content. Based on the results of Srivastave (1978), the following equation is deduced:

$$NC_{rr}=4\times10^{-8}N_r^2\ \lambda_r^2\ \rho[-\exp(-0.15\lambda_r)+S_n\exp(-0.2305\lambda_r)]$$
$$+3.66\times10^{-5}N_r\lambda_r(34-2\lambda_r)[\exp(0.4(34-2\lambda_r))-1], \tag{77}$$

where S_n is the average number of secondary drops formed after one collision-breakup case, its value lies between 3 to 7 and here is taken to be 3.

12. *Aggregation of Ice Crystals*

This process changes only the number concentration of ice, because the ice aggregation (snow flakes) also belongs in ice category in this model. We have

$$NC_{ii}=\frac{1}{2}\int_0^\infty\int_0^\infty\frac{\pi}{4}(D_1+D_2)^2A_{vi}|D_1^{\frac{1}{3}}-D_2^{\frac{1}{3}}|E_{12}N_{oi}^2\exp(\lambda_iD_1-\lambda_iD_2)D_1D_2\rho\left(\frac{P_0}{P}\right)^{a2},$$

$$dD_1dD=\frac{1}{2}\times\frac{\pi}{24}\frac{A_{Vi}}{A_{mi}}\rho E_{ii}N_iQ_i\lambda_i^{-\frac{1}{3}}\left(\frac{P_0}{P}\right)^{a2}KN_{ii}, \tag{78}$$

$$KN_{ii}=\int_0^\infty\int_0^\infty\exp(-D_1+D_2))D_1D_2(D_1+D_2)^2|D_1^{\frac{1}{3}}-D_2^{\frac{1}{3}}|dD_1dD_2\approx7.703,$$

where the aggregation efficiency $\bar{E}_{ii}$ is a gap in knowledge. Based on the experimental results of Rogers (1974) and others (Pruppacher 1978g) $\bar{E}_{ii}$ is assumed to be a function of temperature:

$$\bar{E}_{ii}=0.2\exp[0.35(T-273)]\{1+4\exp[-0.4(T-259)^2]\}. \tag{79}$$

VI. EQUATIONS OF NUCLEATION, MULTIPLICATION AND AUTOCONVERSION

1. *Ice Nucleation*

The number concentration of activated ice nuclei (N_n) is a function of temperature according to Fletcher (1962):

$$N_N=N_{iN}\exp[B_{iN}(273-T)]/\rho. \tag{80}$$

It is not reasonable for nucleation rate to take a constant value in the form of $NP_{Vi}=N_N/\Delta t$ under the given temperature, because Eq. (80) is deduced from results of experiments in cloud chamber, where N_N is the total number of nucleated ice under the given temperature. If the temperature keeps unchanged in a cloud chamber, the number of nucleated ice will not increase significantly after some time. Besides, it is difficult to choice the value of Δt appropriate to physical reality. According to the conditions of the experiments in cloud chamber, the nucleation rate is taken to be a function of the change rate of temperature:

$$NP_{vi}=\frac{\partial N_N}{\partial T}\ \frac{dT}{dt}, \tag{81}$$

N_N is also a function of vapor supersaturation (S) according to Huffman (1973), and has the form

$$N_N=c\cdot S^K, \tag{82}$$

where K lies in 3—8 and $K=5$ is taken in the model. S is the supersaturation with respect to the ice, $S=(Q_V-Q_{si})/Q_{si}$. Eq. (80) is summarized from experiments in cloud chambers with vapor saturated to the water, $Q_V=Q_{Sw}$, and $S=(Q_{SW}-Q_{si})/Q_{si}$. The ice phase is not stable and nucleation rate is zero, if $Q_V-Q_{si}\leqslant0$. Thus

$$N_{pvi}\begin{cases}=0, & \text{if}\ \frac{dT}{dt}\geqslant0\ \text{or}\ Q_V\leqslant Q_{si}\\ =-N_{iN}B_{iN}\exp[B_{iN}(273-T)]/\rho\left(\frac{Q_v-Q_{si}}{Q_{sw}-Q_{si}}\right)^K\frac{dT}{dt}, & \text{if}\ \frac{dT}{dt}<0\ \text{and}\ Q_V>Q_{si},\end{cases} \tag{83}$$

where $dT/dt \approx w\,\partial T/\partial z$ is taken in the calculation, and

$$P_{V_1} = NP_{V_i} Q_{Vo}. \tag{84}$$

2. *Ice Multiplication*

There are several processes of ice multiplication. The riming multiplication is more important and well-studied. According to Mossop (1976) secondary ice particles are produced when the graupel collects the cloud droplets with diameter greater than $D=24$ μm, in the temperature range -3—$-8°$C. This process is most effective in $T=268$ K at which secondary ice can be produced in average per 250 large cloud droplets impacting on the riming ice surface.

$$NP_{ci} = \frac{A(T)}{250}\int_0^\infty\int_{D*2}^\infty N_{og}\exp(-\lambda_g D_g)\ \frac{\pi}{4} D_g^2 A_{Vg} D_g^{0.8} \bar{E} \rho N_{oc} D_c^2 \exp$$
$$\exp\times(-\lambda_C D_C)\,dD_c dD_g$$
$$= \frac{A(T)}{250} C_{cg} \frac{N_C}{Q_C} \exp(-\beta_2)\left(1+\beta_2+\frac{1}{2}\beta_2^2\right), \tag{85}$$

where

$$A(T)\begin{cases} =0 & , \quad T>270\text{K or } T<265\text{K} \\ =1-\frac{1}{4}(T-268)^2 & , \quad 270>T>268\text{K} \\ =1-\frac{1}{9}(T-268)^2 & , \quad 268>T>265\text{K}, \end{cases} \tag{86}$$

$$\beta_2 = \lambda_C D_2^* = (10\rho_W \pi N_C)^{1/3} Q_C^{-1} 0.0024 = 1.6\beta_1,$$
$$NP_{ei} = NP_{ci}\cdot Q_{CO}. \tag{87}$$

3. *Autoconversion of Cloud Droplets to Rain Drops*

Based on the numerical simulation results of Berry (1968) the following equation of autoconversion is deduced and reasonable results are gotten in the simulation of convective and stratiform clouds by Hu (1979, 1983, 1985, and 1986).

$$A_{cr}\begin{cases} =0 & , \quad \text{when } F_C\leqslant 1 \\ =J_1\rho^2 Q_c^3[360\rho Q_c+1.20N_b/D_b]^{-1}, & \quad \text{when } F_C>1 \end{cases} \tag{88}$$

where N_b and D_b are the number concentration and dispersion of the initial size distribution of cloud droplets respectively, $J_1=0.25$ is taken, F_c is a parameter indicating the broadness of the droplet size distribution. It is predicted by the following equation:

$$\frac{\partial F_c}{\partial t} = -W\frac{\partial F_c}{\partial z} + k\frac{\partial^2 F_c}{\partial z^2} + \frac{\delta F_c}{\delta t} + \frac{1}{T_1}, \tag{89}$$

where T_1 is the time during which the size distribution of cloud droplets can grow into a broad one with significant number of rain embryos. $F_c\geqslant 1$ is identical with $t\geqslant T_1$ for Berry's box model. Under this condition raindrops are converted from the growing cloud droplet population. According to Berry (1968) T_1 is calculated by the following equation:

$$T_1 = [120\rho Q_C + 1.6N_b/D_b]/\rho^2 Q_C^2, \tag{90}$$
$$NA_{Cr} = A_{cr}/Q_{ro}. \tag{91}$$

4. *Autoconversion of Graupel to Hail*

Hailstone is defined as an ice ball with a diameter greater than $D_{*h}=0.5$ cm. It is important to simulate the hail separately from the graupel, because it has larger fall speed and causes severe damage. We assume that graupels with mass larger than the smallest hailstone will transform into hail. The corresponding threshold diameter of the graupel $D_{*g}=\sqrt[3]{\frac{A_{mh}}{A_{mg}}}\times D_{*h}=0.97cm$ due to the difference of the densities of graupel and hail.

$$A_{gh}=A\int_{D_{*g}}^{\infty}\exp(-\lambda_g D)D^3A_{mg}N_{og}dD$$

$$=A\exp(-\lambda_g D_{*g})\left[1+\lambda_g D_{*g}+\frac{1}{2}(\lambda_g D_{*g})^2+\frac{1}{6}(\lambda_g D_{*g})^3\right]Q_g, \qquad (92)$$

$$NA_{gh}=A\int_{D_{*g}}^{\infty}\exp(-\lambda_g D)N_{og}dD=A\exp(-\lambda_g D_{*g})N_g, \qquad (93)$$

where A is the transformation rate and $A=0.01s^{-1}$ is taken. In fact $\exp(-\lambda_g D_{*g})\left[1+\lambda_g D_{*g}+\frac{1}{2}(\lambda_g D_{*g})^2+\frac{1}{6}(\lambda_g D_{*g})^3\right]$ and $\exp(-\lambda_g D_{*g})$ are the ratios between the mass and number of the graupel with diameter larger than D_{*g} and the total mass and number respectively, denoted as r_m and r_n. Their typical values are listed in Table 1.

Table 1. Typical Values for Ratios r_m and r_n

λD_*	20	15	10	5
r_m for $D\geqslant D_*$	3.6×10^{-6}	2.1×10^{-4}	1.0×10^{-2}	2.6×10^{-1}
r_n for $D\geqslant D_*$	2.3×10^{-9}	3.1×10^{-7}	4.5×10^{-5}	6.7×10^{-3}
r_m/r_n	1553	691	228	39

It shows that the autoconversion rate increases by 5 orders of magnitude when $\lambda_g D_{*g}$ changes from 20 to 5 and the corresponding average diameter of graupel ($\bar{D}_g=1.82\lambda_g^{-1}$) changes from 0.1 to 0.4 cm, and that the change rate of mass is greater than change rate of number, especially when average diameter is small. So the average diameter of graupel decreases after autoconversion, in turn, the autoconversion rate decreases. It means that the autoconversion rate can be raised only if the other microprocesses make the diameter of graupel increase. Therefore this equation is implicitly coupled with the others.

5. *Autoconversion of Ice to Graupel*

Supercooled water is usually abundant in convective clouds. Ice crystals with diameter greater than D_{*i} can grow rapidly through the collection of water drops. Their collection growth rate is usually greater than sublimation growth rate. It is assumed in this model that ice crystals begin to convert into graupels only when their diameter $D\geqslant D_{*i}$. $D_{*i}=0.03$ cm is taken based on the theoretical and experimental results.

The equation of autoconversion of ice to graupel is similar to that of graupel to hail:

$$A_{ig}=0.01\int_{D_{*i}}^{\infty}N_{oi}D_i^3\exp(-\lambda_i D)A_{mi}dD$$

$$=0.01\exp(-\lambda_i D_{*i})Q_i\left[1+\lambda_i D_{*i}+\frac{1}{2}(\lambda_i D_{*i})^2+\frac{1}{6}(\lambda_i D_{*i})^3\right], \tag{94}$$

$$NA_{ig}=0.01\int_{D_{*i}}^{\infty}N_{oi}D_i\exp(-\lambda_i D)\,dD=0.01\exp(-\lambda_i D_{*i})N_i\,. \tag{95}$$

The typical value of $\lambda_i D_{*i}$ lies in 20—5, so that the values in Table 1 are also typical for A_{ig}. The physical considerations of autoconversion of graupel to hail described above are also valid for autoconversion of ice to graupel.

VII. EQUATIONS OF FREEZING AND MELTING

1. *Self-freezing of Raindrops*

There are two theories on self-freezing of water drops, established by Bigg and Mason respectively (Pruppacher 1978 h). Two types of equations can be deduced:

(1)
$$NM_{rg}(\text{Bigg})=\int_0^{\infty}N_{or}\exp(-\lambda_r D)\,\frac{\pi}{6}D^3B_b[\exp(A_bT_s)-1]\,dD \tag{96}$$

$$=\frac{1}{\rho_w}Q_rB_b[\exp(A_bT_s)-1],$$

where B_b and A_b are parameters, $T_s=273-T$, and

$$Mrg(\text{Bigg})=\frac{20Q_r^2}{\rho_wN_r}B_b[\exp(A_bT_s)-1]. \tag{97}$$

Wisner (1972) has deduced an equation similar to Eq. (97), in his one-parameter model $A_b=0.66\ \text{K}^{-1}$ and $B_b=10^{-4}\ \text{cm}^{-3}\text{s}^{-1}$ are taken.

(2)
$$NM_{rg}(\text{Mason})=-\int_0^{\infty}N_{or}\exp(-\lambda_r D)\frac{\pi}{6}D^3B_mA_m\exp(A_mT_s)\frac{dT}{dt}\,dD$$

$$=-\frac{Q_r}{\rho_w}B_mA_m\exp(A_mT_s)\frac{dT}{dt}, \tag{98}$$

$$M_{rg}(\text{Mason})=\frac{20Q_r^2}{\rho_wN_r}B_mA_m\exp(A_mT_s)\frac{dT}{dt}. \tag{99}$$

Cotton (1972) has deduced an equation similar to Eq. (99) but in more complex form. He takes $A_m=0.8\ \text{k}^{-1}$ and $B_m=1.67\times10^{-5}\text{cm}^{-3}$ based on the experimental results for double-distilled water. The freezing rate is assumed to be independent of the temperature change rate (dT/dt) according to Bigg, but it is in proportion to $(-dT/dt)$ according to Mason. Experiments have shown that when the temperature of water drops maintains unchanged for a long time after its decrease, the freezing rate is neither unchanged(as predicted by Bigg) nor decreased suddenly to zero (as predicted by Mason), but decreases obviously, say, to 10% of initial value in 10 min. Based on these results a correction is added to Eq.(98):

$$NM_{rg}\begin{cases}=\dfrac{Q_r}{\rho_w}B_mA_m\exp[A_m(T_s-3.2)]\left(-\dfrac{dT}{dt}\right)^{1-0.6A_m}, & \text{for } \dfrac{dT}{dt}\leqslant0\,. \\ =0\,, & \text{for } \dfrac{dT}{dt}>0\end{cases} \tag{100}$$

$$M_{rg}=20\frac{Q_r}{N_r}NM_{rg}, \tag{101}$$

where $A_m=0.6k^{-1}$ and $B_m=8\times10^{-3}\mathrm{cm}^{-3}$ are taken in this model based on the experimental data of Vali (1971) for water of natural rain and hail. Values calculated by Eq. (101) are usually between those by Eqs. (97) and (99).

2. *Melting of Ice*

If ice is approximately taken to have the shape of a disk, then

$$M_{ir}=\int_0^\infty N_{oi}D\exp(-\lambda_i D)\frac{2\pi C}{L_f}[k_t(T-T_o)+L_v k_d\rho(Q_v-Q_{so})]$$

$$\left(1+0.23\sqrt{\frac{A_{vi}\rho}{\mu}}D^{2/3}\right)dD+\frac{C_w}{L_f}C_{ci}(T-T_0),$$

$$M_{ir}=\frac{4}{L_f}[k_t(T-T_0)+L_v k_d\rho(Q_v-Q_{so})]N_i(6A_{mi}N_i/Q_i)^{-1/3}$$

$$\left[1+0.23\Gamma\left(2\frac{2}{3}\right)\sqrt{\frac{A_{vi}}{\mu}}\sqrt{\rho}\,(6A_{mi}N_i/Q_i)^{-1/3}\right]+\frac{C_w}{L_f}C_{ci}(T-T_0), \quad (102)$$

$$NM_{ir}=M_{ir}N_i/Q_i\ . \quad (103)$$

3. *Melting of Graupel*

$$M_{gr}=\int_0^\infty N_{og}\exp(-\lambda_g D)\frac{2\pi D}{L_f}[k_t(T-T_0)+L_v k_d\rho(Q_v-Q_{so})]$$

$$\times(1+0.23Re^{1/2})\,dD+\frac{C_w}{L_f}(C_{cg}+C_{rg})(T-T_0)$$

$$=\frac{2\pi}{L_f}[k_t(T-T_0)+L_v k_d\rho(Q_v-Q_{s0})]N_g(6A_{mg}N_g/Q_g)^{-1/3}$$

$$\times\left[1+0.23\sqrt{\frac{A_{vg}\rho}{\mu}}\,\Gamma(2.9)(6A_{mg}N_g/Q_g)^{-0.3}\right]+\frac{C_w}{L_f} \quad (104)$$

$$\times(C_{cg}+C_{rg})(T-T_0),$$

$$NM_{gr}=M_{gr}N_g/Q_g. \quad (105)$$

4. *Melting of Hail*

$$M_{hr}=\int_{D_*}^\infty N_{oh}\exp(-\lambda_h D)\frac{2\pi D}{L_f}[k_t(T-T_0)+L_v k_d\rho(Q_v-Q_{so})]$$

$$\times0.29\sqrt{\frac{A_{vh}\rho}{\mu}}D^{0.9}dD+\frac{C_w}{L_f}(C_{ch}+C_{rh})(T-T_0)$$

$$\approx\frac{0.58\pi}{L_f}\sqrt{\frac{A_{vh}}{\mu}}\sqrt{\rho}\,[k_T(T-T_0)+L_v k_d\rho(Q_v-Q_{s0})]N_h\lambda_h^{-1.9}$$

$$\times[(\lambda_h D_*)^{1.9}+1.8274(0.9\lambda_h D_*+1)]+\frac{C_w}{L_f}(C_{ch}+C_{rh})(T-T_0), \quad (106)$$

$$NM_{hr}=M_{hr}N_h/Q_h. \quad (107)$$

The number change rate due to melting can be deduced under several assumptions: (A) In one-parameter model, N_0 is assumed to be constant and λ_h to increase due to melting. Thus the number of large hailstones would decrease faster than the small ones, which is contrary to fact. (B) λ_h is assumed to decrease and the number concentration of hailstones keeps unchanged at certain diameter (D_{00}). It means that the number concentrations of hailstones with diameter greater than D_{00} would increase due to melting. So it cannot be accepted either. (C) λ_h is assumed to be constant, then $NM_{hr}/N_h=M_{hr}/Q_r$.

5. *Threshold Value of Wet Growth Regime of Hail*

$$C_{hw}=\Big\{-\int_{D_*}^{\infty} N_{oh}\exp(-\lambda_h D)2\pi D^{1.9}0.29\sqrt{\frac{A_{vh}}{\mu}}\sqrt{\rho}[k_T(T-T_0)+L_v k_d\rho$$

$$\times(Q_v-Q_{so})]dD-C_{ih}C_i(T-T_0)\Big\}/[L_f+C_w(T-T_0)]$$

$$\approx-\Big\{0.58\pi\sqrt{\frac{A_{vh}}{\mu}}\cdot\sqrt{\rho}[k_T(T-T_0)+L_v k_d\rho(Q_v-Q_{s0})]$$

$$\times N_h\lambda_h^{1.9}[(\lambda_h D_*)^{1.9}+1.8274(0.9\lambda_h D_*+1)]+C_{ih}C_i(T-T_0)\Big\}\Big/[L_f+C_w$$

$$\times(T-T_0)], \tag{108}$$

where C_w and C_i are the specific heat of water and ice respectively. When hail grows in the wet growth regime ($C_{ch}+C_{rh}>C_{wh}$), the source terms due to collection of water drops by hail are expressed as follows: $\delta Q_h/\delta t=C_{wh}$, $\delta N_h/\delta t=0$, $\delta Q_r/\delta t=C_{ch}-C_{wh}$, $\delta N_r/\delta t=(C_{rh}+C_{ch}-C_{wh})/Q_{ho}-NC_{rh}$ and , $\delta Q_c/\delta t=-C_{ch}$.

VIII. TEST OF MODEL

The full set of equations in this model consists of all microphysical equations in the form of Eq. (26) with source terms (28)—(37) and equations of cloud dynamics and thermodynamics. The model has been tested for many cases in the one-dimensional framework by using the real radiosonde data as input. Three typical examples are given in part II of this work for the simulation of small cumulonimbus cloud, hailstorm and terrential rain. Calculations agree with observations in many aspects. About 2 min. of CPU time on a DPS-7 computer is needed for simulating 1 hour of cloud life by this model in a one-dimensional framework. A few cases have also been simulated by this model in a two-dimensional framework, with Eq. (26) replaced by

$$\frac{\partial M}{\partial t}=-u_j\frac{\partial M}{\partial x_j}+\frac{\partial}{\partial x_j}K\frac{\partial M}{\partial x_j}+\frac{1}{\rho}\frac{\partial\rho V_m M}{\partial Z}+\frac{\delta M}{\delta t}, \tag{109}$$

where j can be taken as 1 and 2 and u_j is air velocity in the x_j direction. About 6 hours of CPU time on a DPS-7 computer is needed for the simulation of 1 hour cloud life.

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