两相饱和介质中的集中力点源 Green 函数*

樊良本[1]，丁伯阳[2]，朱元林[3]，张 颖[2]

(1.浙江工业大学 土木系，浙江 杭州 310014；
2．中国地震局兰州地震研究所，甘肃 兰州 730000；
3．中国科学院兰州冰川冻土研究所，甘肃 兰州 730000)

摘要：通过对两相饱和介质 Biot 方程的变换，利用 Poisson 方程和 Helmholtz 方程性质，求解得到两相饱和介质在集中力点源作用下的位移场 Green 函数.

主题词：格林函数；点源；波场；两相饱和介质

中图分类号：P315.01；O174 **文献标识码**：A **文章编号**：1000-0844(1999)03-0254-05

0 引言

集中力作用下地震波位移场Green函数是地震波正演问题中的一个重要课题.而集中力作用下的两相饱和介质位移场Green函数解，除了可用于对地震孕育、地震预报过程作进一步的探索外，它还是地球物理勘探、地震工程、动力基础理论和土工动测技术等应用学科领域的一个重要基础理论课题.本文通过两相饱和介质Biot方程变换，利用Poisson方程和Helmholtz方程性质，求解得到两相饱和介质在集中力点源作用下的位移场Green函数.并根据求解结果对波场作了阐述.

1 Biot 方程的变换

对于两相饱和介质，Biot 给出的弹性动位移所符合的动力学方程可写为[1]：

$$\begin{cases} \mu\nabla\mathbf{u}_s+\nabla\left[\left(\lambda_c+\mu\right)\nabla\cdot\mathbf{u}_s+\alpha M\nabla\cdot\mathbf{u}_f\right]=\rho\dfrac{\partial^2\mathbf{u}_s}{\partial t^2}+\rho_f\dfrac{\partial^2\mathbf{u}_f}{\partial t^2} \\ \nabla\left(\alpha M\nabla\cdot\mathbf{u}_s+M\nabla\cdot\mathbf{u}_f\right)=\rho_f\dfrac{\partial^2\mathbf{u}_s}{\partial t^2}+\gamma\left(\omega\right)\dfrac{\partial^2\mathbf{u}_f}{\partial t^2} \end{cases} \tag{1}$$

$$\lambda_c=\lambda+\alpha^2M$$

式中：λ和μ是Lame系数，α和M是Biot在两相饱和介质研究中引入的参量[2]；$\mathbf{u}_s$和$\mathbf{u}_f$分别为固相物质和液相相对渗流位移矢量；ρ和ρ_f为固相与流相物质密度；$\gamma(\omega)$是反映Biot问题而引入的系数.

$$\gamma(\omega)=m(\omega)+i\frac{\eta}{\omega k(\omega)} \tag{2}$$

收稿日期：1999-04-13
* 中国科学院兰州冰川冻土研究所冻土工程国家重点实验室资助课题.
作者简介：樊良本(1944—)，男，副教授，主要从事岩土工程学研究.

其中: $k(\omega)$ 为动态渗透率张量, η 为粘滞系数. 而

$$m(\omega) = \alpha(\omega)\frac{\rho_f}{\beta} \tag{3}$$

其中: $\alpha(\omega)$ 为动态空隙弯曲度.

由式(1)能得到以下 2 式:

$$\begin{cases} \left(\lambda_c + \alpha M\xi_1\right)\nabla^2\nabla\circ\mathbf{u}_1 + \left(\lambda_c + \alpha M\xi_2\right)\nabla^2\nabla\circ\mathbf{u}_2 \\ \qquad = \nabla\circ\left[(\rho+\rho_f\xi_1)\dfrac{\partial^2\mathbf{u}_1}{\partial t^2} + \left(\rho+\rho_f\xi_2\right)\dfrac{\partial^2\mathbf{u}_2}{\partial t^2}\right] \\ \mu[\nabla^2\nabla\times\mathbf{u}] = \left(\rho - \dfrac{\rho_f^2}{\gamma(\omega)}\right)\nabla\times\dfrac{\partial^2\mathbf{u}}{\partial t^2} \end{cases} \tag{4}$$

式(4)中

$$\begin{cases} \mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u} \\ \xi_1\mathbf{u}_1 + \xi_2\mathbf{u}_2 = \mathbf{w} \\ \xi_i = \dfrac{\lambda_c + 2\mu - \rho\alpha_i^2}{\rho_f\alpha_i^2 - \alpha M} \qquad (i = 1, 2) \end{cases} \tag{5}$$

$$v_i^2 = \alpha_i^2 = \frac{\triangle \pm \sqrt{\triangle^2 - 4\left(\rho\gamma - \rho_f^2\right)\left[\left(\lambda_c + 2\mu\right)M + \alpha^2M^2\right]}}{2\left(\rho\gamma - \rho_f^2\right)} \qquad (i = 1, 2)$$

$$\triangle = M\rho + \gamma(\lambda_c + 2\mu) - 2\alpha M\rho_f$$

式中: α_1 和 α_2 为快纵波速度与慢纵波速度, v_i 的 4 个根应为两类纵波的正反两个不同方向.

而剪切波速

$$\beta = \sqrt{\frac{\mu}{\rho - \dfrac{\rho_f^2}{\gamma(\omega)}}}$$

这样式(5)实际可以写成

$$\begin{cases} \dfrac{1}{\rho + C\xi_2}\alpha_1^2\nabla^2\nabla\circ\mathbf{u}_1 + \dfrac{1}{\rho + C\xi_1}\alpha_2^2\nabla^2\nabla\circ\mathbf{u}_2 \\ \qquad = \nabla\circ\left[\dfrac{1}{\rho + C\xi_2}\dfrac{\partial^2\mathbf{u}_1}{\partial t^2} + \dfrac{1}{\rho + C\xi_1}\dfrac{\partial^2\mathbf{u}_2}{\partial t^2}\right] \\ \beta^2\nabla^2\nabla\times\mathbf{u} = \nabla\times\dfrac{\partial^2\mathbf{u}}{\partial t^2} \end{cases} \tag{6}$$

2　集中力 $F(t)$ 作用的位移 Green 函数

若有 $F(t)$ 的集中力作用于两相饱和介质 $\boldsymbol{x} = \zeta$ 上, 则可定义集中力的大小 $F(t) = F_0g(t)$, 而

$$F(t) = F_0g(t)[\boldsymbol{Y} + \boldsymbol{Z}]\ \delta(\boldsymbol{x} - \zeta) \tag{7}$$

其中: $\boldsymbol{Y}$ 和 $\boldsymbol{Z}$ 分别应为单位矢量力 $\boldsymbol{K}$ 的无旋场分量和无散场分量. 因此在有体力的情况下, 式(6)应该是

$$\begin{cases} \frac{1}{\rho+\rho_f\xi_2}\alpha_1^2\nabla^2\nabla\cdot\mathbf{u}_1+\frac{1}{\rho+\rho_f\xi_1}\alpha_2^2\nabla^2\nabla\cdot\mathbf{u}_2+F_0g(t)\delta(\boldsymbol{x}-\zeta)Y/\left(\rho+\rho_f\xi_1\right)\left(\rho+\rho_f\xi_2\right) \\ \qquad =\nabla\cdot\left[\frac{1}{\rho+\rho_f\xi_2}\frac{\partial^2\mathbf{u}_1}{\partial t^2}+\frac{1}{\rho+\rho_f\xi_1}\frac{\partial^2\mathbf{u}_2}{\partial t^2}\right] \\ \beta^2\nabla^2\nabla\times\mathbf{u}+F_0g(t)\delta(\boldsymbol{x}-\zeta)\boldsymbol{Z}\Big\backslash\left(\rho-\frac{\rho_f^2}{\gamma(\omega)}\right)=\nabla\times\frac{\partial^2\mathbf{u}}{\partial t^2} \end{cases} \tag{8}$$

对上述方程作 Fourier 变换,得到频率域的方程:

$$\begin{cases} \frac{1}{\rho+\rho_f\xi_2}\alpha_1^2\nabla^2\nabla\cdot\mathbf{u}_1+\frac{1}{\rho+\rho_f\xi_1}\alpha_2^2\nabla^2\nabla\cdot\mathbf{u}_2+\frac{\omega^2}{\rho+\rho_f\xi_2}\nabla\cdot\mathbf{u}_1+\frac{\omega^2}{\rho+\rho_f\xi_2}\nabla\cdot\mathbf{u}_2 \\ \qquad =-F_0g(\omega)\delta(\boldsymbol{x}-\zeta)Y/(\rho+\rho_f\xi_1)(\rho+\rho_f\xi_2) \\ \beta^2\nabla^2\nabla\times\mathbf{u}+\omega^2\nabla\times\mathbf{u}=-F_0g(\omega)\delta(\boldsymbol{x}-\zeta)\boldsymbol{Z}\Big\backslash\left(\rho-\frac{\rho_f^2}{\gamma(\omega)}\right) \end{cases} \tag{9}$$

上式中记 $F(t)$ 的 Fourier 变换 $F(\omega)=F_0g(\omega)$,同时对式(9)注意到 $\boldsymbol{Z}$ 是无散场分量,因此有[3]

$$\boldsymbol{Z}\delta(\boldsymbol{x}-\zeta)=-\frac{\boldsymbol{Z}}{4\pi}\nabla^2\left(\frac{1}{r}\right)=\frac{1}{4\pi}\left[\nabla\times\nabla\times\frac{\boldsymbol{Z}}{r}\right] \tag{10}$$

其中 r 为矢径的模,即是源点到场点的距离.

鉴此,可设方程无散场分量解为

$$\mathbf{u}_2=-F(\omega)\left[\nabla\times\nabla\times(\mathbf{Z}S_\beta)\right] \tag{11}$$

式中的 S_β 满足方程

$$\nabla^2S_\beta+k_\beta^2S_\beta=\frac{1}{4\pi\beta^2r}\cdot\frac{1}{\rho-\frac{\rho_f^2}{\gamma(\omega)}} \tag{12}$$

而 $k_\beta=\frac{\omega}{\beta}$,且 $S_\beta(0,\omega)=0$. 这样

$$S_\beta=\frac{1}{4\pi\omega^2}\left(\frac{1-e^{-ik_\beta r}}{r}\right) \tag{13}$$

再注意到 $Y(t)$ 为无旋场分量,则

$$Y\delta(\boldsymbol{x}-\zeta)=-\frac{1}{4\pi}\left[\nabla\nabla\cdot\left(\frac{Y}{r}\right)\right] \tag{14}$$

方程的无旋场分量解为

$$\mathbf{u}_Y=F(\omega)\left[\lambda_1\nabla\nabla\cdot(YS_{\alpha_1})+\lambda_2\nabla\nabla\cdot(YS_{\alpha_2})\right] \tag{15}$$

$$\lambda_1=\frac{1+\xi_1}{\xi_1-\xi_2};\quad \lambda_2=-\frac{1+\xi_2}{\xi_1-\xi_2}$$

同样

$$\begin{cases} \nabla^2S_{\alpha_1}+k_{\alpha_1}^2S_{\alpha_1}=\frac{1}{4\pi\alpha_1^2r}\cdot\frac{1}{\rho+\rho_f\xi_1} \\ \nabla^2S_{\alpha_2}+k_{\alpha_2}^2S_{\alpha_2}=\frac{1}{4\pi\alpha_2^2r}\cdot\frac{1}{\rho+\rho_f\xi_2} \end{cases} \tag{16}$$

这里 $$k_{\alpha_1}=\frac{\omega}{\alpha_1},\quad k_{\alpha_2}=\frac{\omega}{\alpha_2},\quad S_{\alpha_1}(0,\omega)=0,\quad S_{\alpha_2}(0,\omega)=0$$

所以

$$S_{\alpha_1}+S_{\alpha_2}=\frac{1}{4\pi}\left[\frac{\lambda_1}{\rho+\rho_f\xi_1}\circ\frac{1}{\alpha_1^2}\left(\frac{1-e^{-ik_{\alpha_1}r}}{r}\right)+\frac{\lambda_2}{\rho+\rho_f\xi_2}\circ\frac{1}{\alpha_2^2}\left(\frac{1-e^{-ik_{\alpha_2}r}}{r}\right)\right]\tag{17}$$

由于 $$\left(\nabla^2+k_c^2\right)\frac{e^{-ik_{\alpha_1}r}}{r}=-4\pi\,\delta(\boldsymbol{x}-\zeta)\tag{18}$$

所以方程(9)的解为

$$\mathbf{u}=-\frac{F_0}{4\pi\omega^2}\left\{\frac{\lambda_1}{\rho+\rho_f\xi_1}\left[\nabla\nabla\circ\left(\boldsymbol{I}\frac{e^{-ik_{\alpha_1}r}}{r}\right)\right]+\frac{\lambda_2}{\rho+\rho_f\xi_2}\left[\nabla\nabla\circ\left(\boldsymbol{I}\frac{e^{-ik_{\alpha_2}r}}{r}\right)\right]\right.$$
$$\left.-\frac{1}{\rho-\rho_f^2\backslash\gamma(\omega)}\left[\nabla\times\nabla\times\left(\boldsymbol{I}\frac{e^{-ik_\beta r}}{r}\right)\right]\right\}=F(\omega)\boldsymbol{G}(\boldsymbol{x}/\zeta,\omega)\circ\boldsymbol{K}\tag{19}$$

式中: $\boldsymbol{I}$ 为二阶单位张量, $\boldsymbol{G}(\boldsymbol{x}/\zeta,\omega)$为二阶 Green 函数张量, $\boldsymbol{x}$ 为场点, ζ 为源点. 这里应该是位移 Green 函数 Fourier 谱的形式

$$\boldsymbol{G}(\boldsymbol{x}/\zeta,\omega)=\frac{1}{4\pi\omega^2}\left\{\frac{1}{\rho-\rho_f^2\backslash\gamma(\omega)}\left[\nabla\times\nabla\times\left(\boldsymbol{I}\frac{e^{-ik_\beta r}}{r}\right)\right]\right.$$
$$\left.-\frac{\lambda_1}{\rho+\rho_f\xi_1}\left[\nabla\nabla\circ\left(\boldsymbol{I}\frac{e^{-ik_{\alpha_1}r}}{r}\right)\right]-\frac{\lambda_2}{\rho+\rho_f\xi_2}\left[\nabla\nabla\circ\left(\boldsymbol{I}\frac{e^{-ik_{\alpha_2}r}}{r}\right)\right]\right\}\tag{20}$$

对于单相介质 $\rho_f=0$, 显然式(20)变为

$$\mathbf{u}=-\frac{F(\omega)}{4\pi\omega^2\rho}\left\{\nabla\nabla\circ\left(\boldsymbol{I}\frac{e^{-ik_{\alpha_1}r}}{r}\right)-\nabla\times\nabla\times\left(\boldsymbol{I}\frac{e^{-ik_\beta r}}{r}\right)\right\}\circ\boldsymbol{K}\tag{21}$$

这就是单相介质在集中力 $F(t)$ 作用时的位移谱, 其 Green 函数即是:

$$\boldsymbol{G}(\boldsymbol{x}/\zeta,\omega)=-\frac{1}{4\pi\omega^2\rho}\left\{\nabla\nabla\circ\left(\boldsymbol{I}\frac{e^{-ik_\alpha r}}{r}\right)-\nabla\times\nabla\times\left(\boldsymbol{I}\frac{e^{-ik_\beta r}}{r}\right)\right\}\tag{22}$$

3　集中力 F(t)作用于两相饱和介质时位移场 Green 函数的几种形式

为了运算方便, 下面给出集中力 F(t)作用于两相饱和介质时位移场 Green 函数的几种形式.

$$\boldsymbol{G}(\boldsymbol{x}/\zeta,t)=\frac{1}{4\pi r}\left\{\frac{\mathbf{r}\circ\mathbf{r}}{r^2}\left[\frac{\lambda_1}{\rho+\xi_1\rho_f}\circ\frac{1}{\alpha_1^2}g\left(t-\frac{r}{\alpha_1}\right)-\frac{1}{\rho-\dfrac{\rho_f}{\gamma(\omega)}}\frac{1}{\beta^2}g\left(t-\frac{r}{\beta}\right)\right.\right.$$
$$\left.+\frac{\lambda_2}{\rho+\xi_2\rho_f}\frac{1}{\alpha_2^2}g\left(t-\frac{r}{\alpha_2}\right)\right]+\frac{\boldsymbol{I}}{\beta^2}\frac{1}{\rho-\dfrac{\rho_f^2}{\gamma(\omega)}}g\left(t-\frac{r}{\beta}\right)+\left[3\frac{\mathbf{r}\circ\mathbf{r}}{r^4}-\frac{\boldsymbol{I}}{r^2}\right]$$
$$\left.\left[\frac{\lambda_1}{\rho+\xi_1\rho_f}\int_{r/\alpha_1}^{\infty}g(t-\tau)\tau\,d\tau-\frac{1}{\rho-\dfrac{\rho_f^2}{\gamma(\omega)}}\int_{r/\beta}^{\infty}g(t-\tau)\tau\,d\tau+\frac{\lambda_2}{\rho+\xi_2\rho_f}\int_{r/\alpha_2}^{\infty}g(t-\tau)\tau\,d\tau\right]\right\}\tag{23}$$

注意到 $\nabla\nabla(\frac{1}{r})=\frac{3\mathbf{r}\circ\mathbf{r}}{r^5}-\frac{\boldsymbol{I}}{r^3}$, 上式的分量形式为

$$G_j^i(\boldsymbol{x}/\zeta,\ t)=\frac{1}{4\pi}\left\{\frac{1}{r}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\left[\frac{\lambda_1}{\rho+\xi_1\rho_f}\frac{1}{\alpha_1^2}g\left(t-\frac{r}{\alpha_1}\right)\frac{1}{\rho-\rho_f^2\backslash\gamma(\omega)}\frac{1}{\beta^2}g\left(t-\frac{r}{\beta}\right)\right.\right.$$
$$+\frac{\lambda_2}{\rho+\rho_f\xi_2}\frac{1}{\alpha_2^2}g\left(t-\frac{r}{\alpha_2}\right)\Bigg]+\delta_{ij}\frac{1}{\beta^2 r}g\left(t-\frac{r}{\beta}\right)+\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{1}{r}\right)\Bigg[\frac{\lambda_1}{\rho+\rho_f\xi_1}$$
$$\left.\left.\int_{r/\alpha_1}^{\infty}g(t-\tau)\tau\mathrm{d}\tau-\frac{1}{\rho-\rho_f^2\backslash\gamma(\omega)}\int_{r/\beta}^{\infty}g(t-\tau)\tau\mathrm{d}\tau-\frac{\lambda_2}{\rho+\xi_2\rho_f}\int_{r/\alpha_2}^{\infty}g(t-\tau)\mathrm{d}\tau\right]\right\}\tag{24}$$

或者写成更简洁的形式

$$G_j^i(\boldsymbol{x}/\zeta,\ \omega)=\frac{1}{4\pi}\left\{\frac{1}{\rho-\rho_f^2\backslash\gamma(\omega)}\frac{1}{\beta^2}\delta_{mn}\frac{\partial}{\partial\xi_m}\delta_{nkl}\frac{\partial}{\partial\xi_k}\frac{e^{-ik_\beta r}}{r}\delta_{jl}\right.$$
$$\left.-\frac{\lambda_1}{\rho_s+\xi_1\rho_f}\frac{1}{\alpha_1^2}\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_l}\frac{e^{-ik_{\alpha_1}r}}{r}\delta_{lj}-\frac{\lambda_2}{\rho_s+\xi_2\rho_f}\frac{1}{\alpha_2^2}\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_l}\frac{e^{-ik_{\alpha_2}r}}{r}\delta_{lj}\right\}$$
$$=(\delta_{ik}\delta_{mj}-\delta_{ij}\delta_{mk})S_{mk}-(A_1+A_2)\delta_{ij}\tag{25}$$

其中:

$$S=\frac{1}{\rho-\dfrac{\rho_f^2}{\gamma(\omega)}}\frac{1}{\beta^2}\frac{e^{-ik_\beta r}}{r};\quad A_1=\frac{\lambda_1}{\rho+\xi_1\rho_f}\frac{1}{\alpha_1^2}\frac{e^{-ik_{\alpha_1}r}}{r};\quad A_2=\frac{\lambda_2}{\rho+\xi_2\rho_f}\frac{1}{\alpha_2^2}\frac{e^{-ik_{\alpha_2}r}}{r}$$

[参考文献]

[1] 刘银斌, 李幼铭, 吴如山, 等. 横向各向同性多孔介质中的地震波传播[J]. 地球物理学报, 1994, 37(4): 499~514.

[2] Biot M A. Theory of propagation of elastic wave in a fluid-saturated soil[J]. J Acoust S A, 1956, 28(162): 168~178.

[3] Aki K, Richards P G. Quantitative Seismology[M]. W H Freeman Comp, 1980.

[4] Nur A. Dilatency pore fluid and premonitory variations of t_p/t_s travel times[J]. B S S A, 1972, 62(5): 381~410.

THE GREEN FUNCTION ON TWO-PHASE SATURATED MEDIUM BY CONCENTRATED FORCE

FAN Liang-ben[1], DING Bo-yang[2], ZHU Yuan-lin[3], ZHANG Ying[2]

(1. *Zhejiang Industrial University*, *Hangzhou* 310014, *China*;

2. *Lanzhou Institute of Seismology*, *CSB*, *Lanzhou* 730000, *China*;

3. *SKLFSE*, *Lanzhou Institute of Glaciology and Geocryology*, *CAS*, *Lanzhou* 730000, *China*)

Abstract

The Green function on two-phase saturated medium by concentrated force is given and the wave field is discussed according to a Biot equation.

Key words: Green function; Concentrated force; Wave field; Two-phase saturated medium