THE INSTABILITY OF TRAJECTORY AND ERROR GROWTH *

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ABSTRACT

The purpose of this paper is to study the dynamical mechanism of error growth in the numerical weather prediction. The error is defined in the sense of generalized energy, simply called energy error. From the spectral form of the primitive equations, we have derived the evolution equations of error in detail. The analyses of these equations have shown that the error growth rate is determined by the tangent linear equations. The nonlinear advection caused by the error perturbation itself contributes nothing to the error growth rate, and only redistributes the error. Furthermore, an approach to calculation of the error growth rate has been developed, which can also be used to study the local instability of time–independent basic state as well as time–dependence basic state. This approach is applied to well–known Lorenz's system, and the results are indicative of the correctness and significance of the theoretical analyses.

Key words: energy error, error growth rate, local trajectory instability, tangent linear equation

I. INTRODUCTION

Up to now, no clear dynamical mechanism has been proposed to explain the continual growth of the initial error in the numerical weather prediction (NWP). Early in 1963, Lorenz revealed that the general behavior of a deterministic system might be nonperiodic and nonperiodic variation is essentially sensitive to the error in initial state. Therefore, the notion of predictability has been introduced (Lorenz 1965). Because of the continual growth of the error in initial state, two states differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states within a finite time interval. Evidently, the NWP can not provide useful information beyond this time interval, i.e., unpredictability. Many previous studies aimed at quantitatively determining the upper limit of the range of theoretical predictability (Charney et al. 1966; Smagrinsky 1963; Leith 1965) and the lower limit of the range of realistic predictability for the atmospheric prediction model (Lorenz 1982). These studies have shown that the general range of predictability is about two weeks with present–day accuracy in the observing state of the atmosphere. Because the growth rate of error is in agreement with the baroclinic instability of normal mode to some extent, it is suggested that the error growth is caused by the baroclinic instability (Lorenz 1981). In addition, the nonlinear processes are emphasized. Chou (1986) has proposed an important type of the mechanism of error growth. When the controllable parameters vary across some special points in the phase parameter space, called bifurcation points, the small error in initial state can lead the final state to go into an attraction basin of the incorrect attractor. These studies have increased our

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understanding of underlying physics of error growth, but have yet not given a clear physical or dynamical depiction.

The recent studies on predictability have provided us with two new facts: first, increase of the resolution and complexity of the numerical prediction model causes decrease of the upper limit of the range of theoretical predictability (Lorenz 1982); second, the range of predictability for a particular prediction highly depends on flow patterns itself, and some special patterns possess much larger range of predictability, for example, blocking high (Gilchrist 1986; Miyakoda et al. 1986) and PNA patterns with the positive index (Palmer 1988). There is still a strong need for dynamical study to give explanation to these facts.

In this study, the evolution equation of error has been derived in detail. The notion of local instability of trajectory has been introduced and an approach for its calculation is developed. The relationship between the error growth and local instability of trajectory is discussed, and the dynamical mechanism of error growth is then described.

II. PROGNOSTIC EQUATION

In Chou's researches (1983; 1986), the dynamic equation governing the large scale atmospheric motion can be written in a simple operator form:

$$B\frac{\partial\varphi}{\partial t}+(N+L)\varphi=\xi, \tag{1}$$

where the state vector $\varphi=(v_\lambda, v_\theta, \omega, \Phi, T)^T$. The superscript "$T$" denotes a transpose, and

$$N=\begin{bmatrix} \Lambda & 2\Omega\cos\theta+\frac{\mathrm{ctg}\theta}{a}v_\lambda & 0 & \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & 0 \\ -2\Omega\cos\theta\frac{\mathrm{ctg}\theta}{a}v_\lambda & \Lambda & 0 & \frac{1}{a}\frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial P} & \frac{R}{P} \\ \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & \frac{\partial}{\partial P} & 0 & 0 \\ 0 & 0 & -\frac{R}{P} & 0 & \frac{R^2}{C^2} \end{bmatrix}, \tag{2}$$

$$L=\begin{bmatrix} -\mu_1\nabla^2-\frac{\partial}{\partial P}v_2(\frac{gp}{RT})^2\frac{\partial}{\partial P} & 0 & 0 & 0 & 0 \\ 0 & -\mu_1\nabla^2-\frac{\partial}{\partial P}v_1(\frac{gp}{RT})^2\frac{\partial}{\partial P} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2\nabla^2-\frac{\partial}{\partial P}v_2(\frac{gp}{RT})\frac{\partial}{\partial P} \end{bmatrix}, \tag{3}$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R^2}{C^2} \end{bmatrix}, \tag{4}$$

$$\Lambda=\frac{v_\lambda}{a\sin\theta}\frac{\partial}{\partial\lambda}+\frac{v_\theta}{a}\frac{\partial}{\partial\theta}+\omega\frac{\partial}{\partial p}, \tag{5}$$

$$\nabla^2=\frac{1}{a^2\sin^2\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{a^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}, \tag{6}$$

$$C^2=\frac{R^2\overline{T}}{g}(\gamma_d-\overline{\gamma}), \tag{7}$$

where $\overline{\Phi}$ and $\overline{T}$ are the spatial means of potential height and temperature respectively, varying only with the pressure p. Φ and T are the deviations with respect to them. ε is diabatic heating rate and we have $\xi=(0, 0, 0, 0, R^2\varepsilon / C^2C_p)^T$. Others are the same as commonly used.

The lower boundary conditions are taken at $p=P$

$$v_\lambda=v_\theta=\omega=0, \tag{8}$$

$$\frac{\partial T}{\partial p}=\alpha_s(T_s-T), \tag{9}$$

where $P=1000$ hPa. The orographic effect has been ignored.

The upper boundary conditions can be written as

$$\lim_{p\to 0}p^2v_0=\lim_{p\to 0}p^2v_\lambda=\lim_{p\to 0}\omega=\lim_{p\to 0}p^2\frac{\partial T}{\partial p}=0. \tag{10}$$

We have chosen an inner product

$$(\varphi_1, \varphi_2)=\int_\Omega\varphi_1^T\varphi_2\mathrm{d}\Omega, \tag{11}$$

where Ω denotes the entire atmosphere. We can get easily

$$(\varphi_1, N\varphi_2)=-(\varphi_2, N\varphi_1), \tag{12}$$

$$(\varphi, N\varphi)=0, \tag{13}$$

$$(\varphi, B\varphi)\geqslant 0, \qquad (\varphi_1, B\varphi_2)=(B\varphi_1, \varphi_2), \tag{14}$$

$$(\varphi, L\varphi)\geqslant 0, \qquad (\varphi_1, L\varphi_2)=(L\varphi_1, \varphi_2). \tag{15}$$

Equations (12)—(15) indicate that N is an anti–adjoint operator, but L and B are positive self–adjoint operators.

It should be pointed out: N varies as a function of φ, and is a nonlinear operator; L is a linear operator; B is a linear and constant operator.

We may let e_1, e_2, e_3, $\cdots$ be a complete orthonormal basis, which satisfies horizontally periodic condition and vertically homogeneous boundary conditions (8)—(10). So a given state vector can be expanded

$$\varphi = \sum_i \varphi_i e_i, \tag{16}$$

where

$$\int_\Omega e_i e_j \mathrm{d}\Omega = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{17}$$

$$\varphi_i = \int_\Omega \varphi e_i \mathrm{d}\Omega, \tag{18}$$

φ_i is the spectral coefficient and also a vector.

Substituting (16) into (1), we have

$$B\frac{\mathrm{d}}{\mathrm{d}t}(\sum_i \varphi_i e_i) + (N+L)\sum_i \varphi_i e_i = \sum_i \xi_i e_i. \tag{19}$$

Using Galerkin approximation, we get

$$B\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} + \Phi_i - \Psi_i = \xi_i, \tag{20}$$

where

$$\Phi_i = < N\sum_i \varphi_i e_i, e_i >, \tag{21}$$

$$\Psi_i = < \sum_i L\varphi_i e_i, e_i > \tag{22}$$

and

$$< f_1, f_2 > = \int_\Omega f_1 f_2 \mathrm{d}\Omega. \tag{23}$$

After some mathematical manipulations, we can obtain the prognostic equation in spectral form:

$$B\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} + \sum_j N_{ij}\varphi_j + \sum_j L_{ij}\varphi_i = \xi_i, \tag{24}$$

where N_{ij} and L_{ij} are matrices corresponding to matrix operators N and L respectively, and we have

$$N_{ij} = (N_{mnij}), \tag{25}$$

$$L_{ij} = (L_{mnij}), \tag{26}$$

N_{mnij} and L_{mnij} are given by

$$N_{mnij} = < N_{mnij}, e_i >, \tag{27}$$

$$L_{mnij} = < L_{mnij}, e_i >. \tag{28}$$

Considering that N is an anti–adjoint operator and L a self–adjoint operator, it is easily shown

$$N_{mnij} = -N_{nmij}, \qquad N_{mnij} = -N_{nmji}, \tag{29}$$

$$L_{mnij} = L_{nmij}, \qquad L_{mnij} = L_{nmji}. \tag{30}$$

It is noteworthy that N_{ij} varies as the function of φ_i, so N_{ij} is a nonlinear operator. But L_{ij} is a linear operator. Eqs.(26) and (27) indicate that the spectral expression of continuous equation (1) does not change the properties of the operator N and L, which is just the advantage of the Galerkin approximation.

III. TRAJECTORY INSTABILITY AND LOCAL INSTABILITY EXPONENT

Supposing that Eq.(1) is truncated at the total mode–number I, and considering that φ is a 5–dimensional vector, an M–dimensional phase space is generated, where $M=I\times J$ and $J=5$. A state of the model atmosphere at future time t is defined by a point in this phase space. The time evolution of the state starting from the initial state $B\varphi_0$ is described by a curve which is called as a trajectory.

Generally, the trajectory C is called stable if a trajectory lying in a sufficiently small neighborhood of the trajectory C initially tends to keep in a small neighborhood of the trajectory C. This definition of instability of a trajectory can be stated mathematically: The trajectory C is called stable if for a given $\varepsilon>0$, there exists $\eta>0$ such that a representative point R of another trajectory lying in the neighborhood η of the trajectory C at time τ keeps in the neighborhood $\eta(\varepsilon)$ of the trajectory C for all $t>\tau$. If there exists no such η, then the trajectory C is said to be unstable. Furthermore, we can define asymptotic stability. The trajectory is called asymptotically stable if it is stable and the distance between point R and trajectory C tends to be zero for $t\to\infty$.

There are two methods to analyze the trajectory instability: local and global methods. The global method requires to find a Lyapunov function, and the complexity of the governing equations for the model atmosphere obstructs the use of this method. The local method is extensively employed, which requires only to find all eigenvalues of the linear part of the governing equation.

In the following, we shall introduce the concept of local instability of trajectory. Consider a trajectory $\overline{\varphi}(t)(\overline{\varphi}_1(t), \overline{\varphi}_2(t), \cdots, \overline{\varphi}_M(t))$, and another trajectory in its neighborhood

$$\varphi_i(t)=\overline{\varphi}_i(t)+\varepsilon_i(t), \tag{31}$$

where $\varepsilon_i(t)$ denotes a small perturbation. From Eq. (24), the evolution of a small perturbation can be determined by the linearized equation

$$B\frac{\mathrm{d}\varepsilon_i}{\mathrm{d}t}+\sum_j\overline{N}_{ij}\varepsilon_j+\sum_j N'_{ij}\overline{\varphi}_j+\sum_j L_{ij}\varepsilon_j=\sum_j F_{ij}\varepsilon_j, \tag{32}$$

where F_{ij} is a matrix, and $F_{ij}=(\partial\xi_i/\partial\overline{\varphi}_j)$.

Considering that B is symmetrical and $\overline{N}_{ij}$ is anti–symmetrical, add Eq.(32) right–multiplied by ε_j^T to the transposed (32) left–multiplied by ε_i:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)+\sum_i\sum_j\left(\varepsilon_i^T N'_{ij}\overline{\varphi}_j+\overline{\varphi}_j^T N'_{ij}\varepsilon_i\right)+2\sum_i\sum_j\left(\varepsilon_i^T L_{ij}\varepsilon_j\right)\\&\quad=\sum_i\sum_j\left(\varepsilon_i^T F_{ij}\varepsilon_j-\varepsilon_j^T F_{ij}^T\varepsilon_i\right).\end{aligned} \tag{33}$$

In fact, $\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)$ is the perturbation energy in some sense, and $\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)$ determines the rate of amplitude change of the perturbation ε. From these, we can give definitions: trajectory is called locally stable at some point $\overline{\varphi}(t)$ in the phase space if $\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)<0$;

trajectory called locally unstable if $\frac{d}{dt}\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)>0$; trajectory called neutral if $\frac{d}{dt}\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)=0$.

Let $x=(\varphi_1, \varphi_2, \cdots, \varphi_I)$, we can obtain

$$-\sum_i\sum_j\left(\varepsilon_i^T N'_{ij}\overline{\varphi}_j\right)=x^T Ax, \tag{34}$$

$$-2\sum_i\sum_j\left(\varepsilon_i^T L_{ij}\varepsilon_j\right)=x^T Dx, \tag{35}$$

$$\sum_i\sum_j\left(\varepsilon_i^T F_{ij}\varepsilon_j\right)=x^T Fx. \tag{36}$$

Thus, Eq.(33) becomes

$$\frac{d}{dt}\|x\|^2=x^T(A+A^T+D+F+F^T)x, \tag{37}$$

where $\|x\|=\sum_i\left(\varepsilon_i^T B\varepsilon_i\right)$. From the symmetry of L_{ij}, it follows that D is symmetrical. Let $H=A+A^T+D+F+F^T$. H is a symmetrical matrix with the order M.

It is a fundamental theorem of linear algebra that the eigenvector of a symmetrical matrix such as H can be chosen to form a complete orthonormal basis and the eigenvalues are all real.

Let $E_1, E_2, \cdots, E_M$ denote the eigenvectors and $\lambda_1, \lambda_2, \cdots, \lambda_M$ the associated eigenvalues, and suppose that the eigenvalues are arranged so that $\lambda_1>\lambda_2>\cdots>\lambda_M$. Any given state vector x can be expanded into the eigenvectors:

$$x=\sum_{i=1}^{M}x_i E_i. \tag{38}$$

Substituting (38) into Eq.(37), we get

$$\frac{d}{dt}\|x\|^2=\sum_{i=1}^{M}\lambda_i x_i^2. \tag{39}$$

Hereafter, we refer to $\lambda_i (i=1, 2, \cdots, M)$ as a characteristic exponent of local instability of trajectory for the phase space point $\overline{\varphi}(t)$, and refer to the sub–space spanned by the eigenvector associated positive eigenvalues as a characteristic sub–space of local instability of trajectory. From (39), we can give the criteria for local instability of trajectory:

(1) Trajectory is locally stable if all eigenvalues are negative.

(2) Trajectory is locally unstable if there exist positive eigenvalues. Furthermore, it is absolutely unstable if eigenvalues are all positive, or conditionally unstable if at least one of eigenvalues is negative.

Here, we can give an estimation of the perturbation amplitude. We get from (39)

$$\frac{d}{dt}\|x\|^2\leqslant\lambda_1\|x\|^2. \tag{40}$$

Thus

$$\|x\|\leqslant\|x_0\|\exp\left(\int_{t^0}^{t}\frac{\lambda_1}{2}d\tau\right). \tag{41}$$

The analyses above provide us with criteria for local instability of trajectory and with the method of finding characteristic exponents. In the following section, we shall indicate that the local trajectory instability determines the growth rate of error.

IV. ERROR EVOLUTION

We start from (24). Consider a trajectory $\overline{\varphi}(t)$ which describes a correct solution for equation (24). The trajectory with error may be written:

$$\overline{\varphi}(t) = \overline{\varphi}(t) + \varepsilon(t), \tag{42}$$

where $\varepsilon(t)$ is the difference between the correct trajectory and the trajectory with error. We can obtain the governing equation for $\varepsilon(t)$

$$B\frac{d\varepsilon_i}{dt} + \sum_j \overline{N}_{ij}\varepsilon_j + \sum_j N'_{ij}\overline{\varphi}_j + \sum_j N'_{ij}\varepsilon_i + \sum_j L_{ij}\varepsilon_i = \xi'_i + \sum_j F_{ij}\varepsilon_j. \tag{43}$$

Comparing (43) with (32), we can see that two new terms have been introduced, i.e., nonlinear term of error $\sum_j N'_{ij}\varepsilon_i$ and term of heating error ξ'_i.

Through the processes similar to that in the previous section, we can obtain the evolution equation of error

$$\frac{d}{dt}\sum_i\left(\varepsilon_i^T B\varepsilon_i\right) + \sum_i\sum_j\left(\varepsilon_i^T N'_{ij}\overline{\varphi}_j + \overline{\varphi}_j^T N'^T_{ij}\varepsilon_i\right) + 2\sum_i\sum_j\left(\varepsilon_i^T L_{ij}\varepsilon_j\right)$$
$$= \sum_i\sum_j\left(\varepsilon_i^T F_{ij}\varepsilon_i - \varepsilon_j^T F_{ij}\varepsilon_i\right) + 2\xi'_i\varepsilon_i^T. \tag{44}$$

The equation

$$\sum_i\sum_j\left(\varepsilon_i^T N'_{ij}\varepsilon_j - \varepsilon_j^T N^T_{ij}\varepsilon_i\right) = 0 \tag{45}$$

has been considered in obtaining (44). Eq.(45) is of importance, which shows that the nonlinear advection caused by the error perturbation itself contributes nothing to the error growth rate, and only redistributes the error.

Compare (44) with (33). If the terms of heating error are ignored, i.e., $\xi'=0$, (44) is the same as (33). It follows from this that it is the local instability of trajectory that determines the growth rate of error and the instability is linear. These conclusions make it possible for us to give a clear physical depiction: The growth rate of error is determined by local instability of trajectory and the given error vector; the error will grow if the error vector possesses a projection on the unstable characteristic subspace, or the error will tend to reduce if the error vector has no projection on the unstable characteristic subspace; the largest growth rate of error is realized when the error vector is set equal to the eigenvector associated with the largest eigenvalue.

It should be pointed out that the local instability of trajectory is a necessary condition but not a sufficient condition for the error growth. This is because the error growth depends not only the local instability of trajectory but also the characteristic of the error vector itself. But the absolute instability is the sufficient condition for error growth.

The results above also suggest that the growth rate of error will vary with the different positions on the trajectory associated with the exact solution. From this, we can explain the fact that the persistent flow patterns possess larger range of predictability. In fact, the persistent flow patterns are more stable. It can also give an explanation for the decrease of the upper limit of the range of theoretical predictability with the increase of the resolution of the numerical weather prediction model. As well–known, the instability strength of a given model atmosphere state strongly depends on the resolution of the model. Higher resolution will allow stronger instability. Therefore, the model with higher resolution will reduce the upper limit of theoretical predictability.

V. APPLICATION TO LORENZ'S SYSTEM

Lorenz's system has been extensively studied (for example, see Chou's review (1990)). In the following, we will discuss the local instability of trajectory when the attractors are both fixed points and chaos, and the effect of local instability of trajectory on the error growth.

1. *Error Evolution and Criterion for Local Instability*

As well–known, equations of Lorenz's system are

$$\frac{dx}{dt} = \sigma y - \sigma x, \tag{46}$$

$$\frac{dy}{dt} = \gamma x - y - xz, \tag{47}$$

$$\frac{dz}{dt} = -bz + xy. \tag{48}$$

Let $X = (x, y, z)^T$, they can be written in an operator form:

$$\frac{dX}{dt} = LX + NX, \tag{49}$$

where

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}. \tag{50}$$

As can be seen, N is anti–symmetrical and nonlinear operator, but L is linear and constant operator. We easily obtain the evolution equation of error:

$$\frac{dX^T X}{dt} = X(L + L^T + B + B^T)X, \tag{51}$$

where

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -\bar{z} & 0 & 0 \\ \bar{y} & 0 & 0 \end{bmatrix}. \tag{52}$$

Equation (48) also indicates that the growth rate of error is only determined by the tangent linear equation.

Let $C = L + L^T + B + B^T$. From the discussion in Section III, it is obvious that the trajectory is locally stable if the eigenvalues of matrix C are all negative or locally instable if at least one of eigenvalues is positive.

2. *In the Attraction Basin of the Attractor at a Fixed Point*

With the analyses of local instability of trajectory which is located in the attraction basin of the attractor at a fixed point, we shall show that the strong local instability of trajectory can occur even in part of the attraction basin of the attractor, so that rapid growth of error can occur.

In fact, the local instability of trajectory will vary complicatedly if there exists more than one attractor at fixed points. Some particular trajectories may be of local instability obviously. For example, the separatrix is locally unstable, which refers to the mechanism for error growth found by Chou (1990). But the local instability can lead to rapid growth of error even within the attraction basin of the attractor, which is strong enough to cause the loss of predictability in some sense. Figure 1a shows the time evolution of the largest instability exponent for $\gamma = 18.0$ (where $\sigma = 10.0$, $b = 8/3$, the same in the following) and initial state (1.0, 2.0, 3.0). For this case, there exist three steady solutions of Lorenz's system, and two of them are stable. It can be seen that the strong local instability occurs within the calculation period. Figure 2b shows the time evolution of the root–mean–square (RMS) difference between two trajectories for initial states (1.0, 2.0, 3.0) and (2.0, 3.0, 4.0). The large error can be found from 2.0 to 5.0 time units, the maximums of which even exceed the RMS of the variation of the trajectory itself. So the trajectory may be considerably sensitive to the initial state in part of the attraction basin of the attractor at a fixed point so that the predictability may lose in some sense.

For $\gamma = 24.0$, we repeat the calculation above. In this case, there still exist two attractors at fixed points. It can be seen from Fig.2a that the instability becomes stronger. The large error can be found first at 5 time units. After this time, the error varies with very large amplitude which does not reduce within a long time. Such variation of error can be well explained only by the local instability of trajectory.

3. *On Chaotic Attractor*

If $\gamma = 28.0$, Lorenz's system goes into chaos, which is called as "a standard Lorenz's attractor". We still take the initial state (1.0, 2.0, 3.0), but the initial state with error is (1.5, 2.5, 3.5). As we expect, the instability becomes much stronger (Fig.3a). From Fig.3b, it can be found

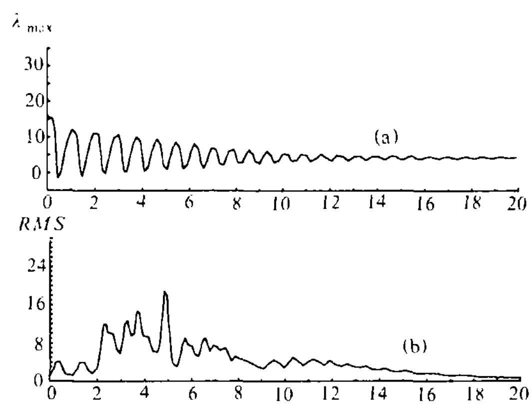

Fig. 1. The time evolutions of (a) the largest instability exponent and (b) root–mean–square error for $\gamma = 18.0$ and initial state (1.0, 2.0, 3.0) (for detailed description, see the text).

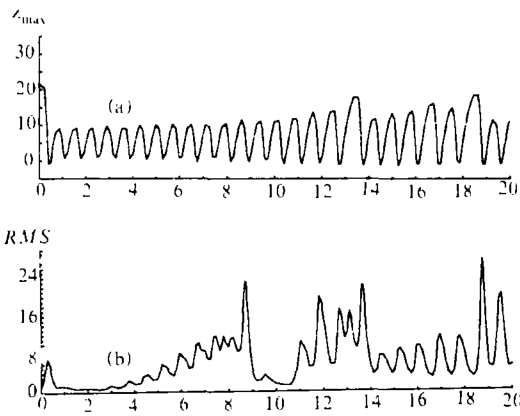

Fig. 2. As in Fig.1 but for $\gamma = 24.0$.

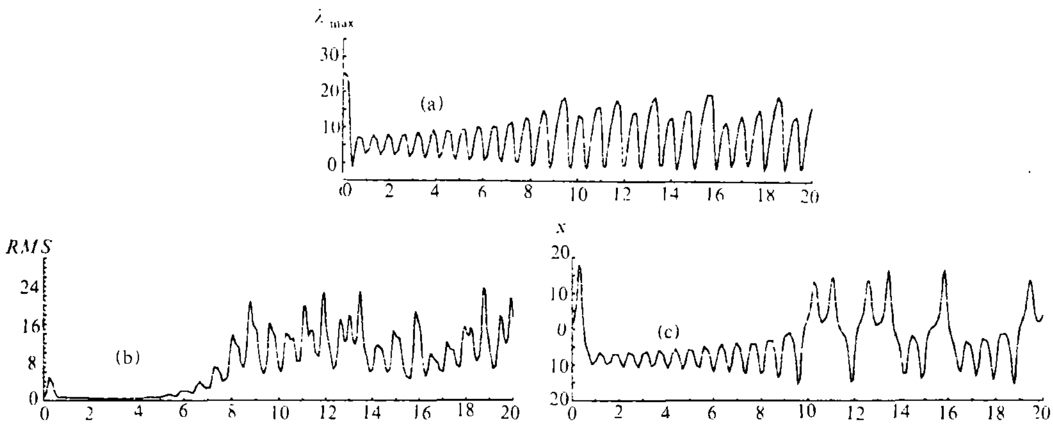

Fig. 3. The time evolutions of the chaotic state for $\gamma=28.0$. (a) The largest local instability exponent; (b) the root–mean–square error; (c) component x.

that the RMS error rapidly grows after 6 time units. It is worthy to point out that the large error, strong local instability and rapid transition of the system state are remarkably consistent.

The analyses above have shown that the local instability of trajectory can well describe the detailed structure of the chaotic attractor, and the growth rate of error varies greatly for the different positions on the chaotic attractor.

VI. CONCLUSIONS AND DISCUSSIONS

Because of the continual growth of the error in initial state, the range of predictability commonly accepted is only about two weeks with present–day accuracy in the observing state of the atmosphere. The extended range forecast beyond this range of predictability naturally requires the deep understanding of the dynamical processes of error growth.

In present study, the error is defined in the sense of generalized energy, simply called energy error. From the spectral form of the primitive equations, we have derived the evolution equations of error in detail. The analyses of these equations have shown that the error growth rate is determined by the tangent linear equations. The nonlinear advection caused by the error perturbation itself contributes nothing to the error growth rate, and only redistributes the error. The local instability of trajectory is the most essential causes for error growth. These results have shown that the growth rate of error highly depends on the features of the atmospheric state itself within the forecast period. From these, it is suggested that new approaches of extended range forecast should be found to obstruct the rapid growth of error associated with the strong local instability of trajectory but not to distort the interesting quantities of extended range forecast.

Furthermore, an approach to calculate the error growth rate has been developed, which can also be used to study the local instability of time–independent basic state as well as time–dependent basic state. This approach has been applied to the well–known Lorenz's system. The results indicate that the trajectory can possess local instability even within the attraction basin of the attractor at a fixed point which may be strong enough to lead to the loss of predictability for some period. On chaotic attractor, the local instability varies considerably with the evolution of system's state. The trajectory can be even locally stable at some positions in the phase space.

The range of predictability will be small if the local instability is strong. In contrast, the range of predictability will be large if the local instability is weak. We think that specially the large range of predictability for some flow pattern is only the reflection of relative weakness of the local instability of trajectory.

It should be pointed out that we have not analyzed the influence of model deficiency on the predictability. This is a very difficult but important problem which strongly requires further researches.

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